Multi-mode bosonic Gaussian channels

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July 8, 2008

Abstract

A complete analysis of multi-mode bosonic Gaussian channels is proposed. We clarify the structure of unitary dilations of general Gaussian channels involving any number of bosonic modes and present a normal form. The maximum number of auxiliary modes that is needed is identified, including all rank deficient cases, and the specific role of additive classical noise is highlighted. By using this analysis, we derive a canonical matrix form of the noisy evolution of n-mode bosonic Gaussian channels and of their weak complementary counterparts, based on a recent generalization of the normal mode decomposition for non-symmetric or locality constrained situations. It allows us to simplify the weak-degradability classification. Moreover, we investigate the structure of some singular multi-mode channels, like the additive classical noise channel that can be used to decompose a noisy channel in terms of a less noisy one in order to find new sets of maps with zero quantum capacity. Finally, the two-mode case is analyzed in detail. By exploiting the composition rules of two-mode maps and the fact that anti-degradable channels cannot be used to transfer quantum information, we identify sets of two-mode bosonic channels with zero capacity.

Contents

Bosonic Gaussian channels are ubiquitous in physics. They arise whenever a harmonic system interacts linearly with a number of bosonic modes which are inaccessible in principle or in practice [1, 2, 3, 4, 5, 6, 7]. They provide realistic noise models for a variety of quantum optical and solid state systems when treated as open quantum systems, including models for wave guides and quantum condensates. They play a fundamental role in characterizing the efficiency of a variety of tasks in continuous-variables quantum information processing [8], including quantum communication [9] and cryptography [10]. Most importantly, communication channels such as optical fibers can to a good approximation be described by Gaussian quantum channels.

Not very surprisingly in the light of the central status of such quantum channels, a lot of effort has been recently devoted to studying their properties (see Ref. [4] for a review), based on a long tradition of work on Gaussian channels [6, 2, 3]. Specifically, from a quantum information perspective, a key question is whether or not a channels allows for the reliable transmission of classical or quantum information [3, 4, 11, 12, 13, 14, 15, 16, 17, 18]. Significant progress has been made in this respect in recent years, although for some important cases, like the thermal noise channel modelling a realistic fiber with offset noise, the quantum capacity is still not yet known. In this context, the degradability properties represent a powerful tool to simplify the quantum capacity issue of such Gaussian channels. Indeed, in Refs. [16, 17] it has been shown that with some (important) exceptions, Gaussian channels which operate on a single bosonic mode (i.e., one-mode Gaussian channels) can be classified as weakly degradable or anti-degradable. This paved the way for the solution of the quantum capacity [19] for a large class of these maps [15].

Here, first we propose a general construction of unitary dilations of multi-mode quantum channels, including all rank-deficient cases. We characterize the minimal noise maps involving only true quantum noise. Then, by using a generalized normal mode decomposition recently introduced in Ref. [20], we generalize the results of Refs. [16, 17] concerning Gaussian weak complementary channels to the multi-mode case giving a simple weak-degradability/anti-degradability condition for such channels. The paper ends with a detailed analysis of the two-mode case. This is important since any n-mode channel can always be reduced to single-mode and two-mode components [20]. We detalize the degradability analysis and investigate a useful decomposition of a channel with the additive classical noise map that allows us to find new sets of channels with zero quantum capacity.

1 Multi-mode bosonic Gaussian channels

Gaussian channels arise from linear dynamics of open bosonic system interacting with a Gaussian environment via quadratic Hamiltonians. Loosely speaking, they can be characterized as CPT maps that transform Gaussian states into Gaussian states [21, 22].

1.1 Notation and preliminaries

Consider a system composed by n bosonic modes having canonical coordinates $\hat{Q}_1, \hat{P}_1, \cdots, \hat{Q}_n, \hat{P}_n$. The canonical commutation relations of the canonical coordinates, $[\hat{R}_j, \hat{R}_{j'}] = i(\sigma_{2n})_{j,j'}$, where $\hat{R} := (\hat{Q}_1, \cdots, \hat{Q}_n; \hat{P}_1, \cdots, \hat{P}_n)$, are grasped by the $2n \times 2n$ *commutation matrix*

$$\sigma_{2n} = \left[\begin{array}{cc} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{array} \right] , \tag{1}$$

when this order of canonical coordinates is chosen, (here $\mathbb{1}_n$ is the $n \times n$ identity matrix) [3, 4, 21]. Even though different reordering of the elements of $\hat{R}$ will not affect the definitions that follow, we find it useful to assume a specific form for σ_{2n}. One defines the group of real *symplectic matrices* $Sp(2n, \mathbb{R})$ as the set of $2n \times 2n$ real matrices S which satisfy the condition

$$S \sigma_{2n} S^T = \sigma_{2n} \, . \tag{2}$$

Since $\mathrm{Det}[\sigma_{2n}] = 1$, and $\sigma_{2n}^{-1} = -\sigma_{2n}$, any symplectic matrix S has $\mathrm{Det}[S] = 1$ and it is invertible with $S^{-1} \in Sp(2n, \mathbb{R})$. Similarly, one has $S^T \in Sp(2n, \mathbb{R})$. Symplectic matrices play a key role in the characterization of bosonic systems. Indeed, define the *Weyl (displacement) operators* as $\hat{V}(z) = \hat{V}^\dagger(-z) := \exp[i\hat{R}z]$ with $z := (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T$ being a column vector of $\mathbb{R}^{2n}$. Then it is possible to show [1] that for any $S \in Sp(2n, \mathbb{R})$ there exists a *canonical* unitary transformation $\hat{U}$ which maps the canonical observables of the system into a linear combination of the operators $\hat{R}_j$, satisfying the condition

$$\hat{U}^\dagger \, \hat{V}(z) \, \hat{U} = \hat{V}(Sz) \, , \tag{3}$$

for all z. This is often referred to as metaplectic representation. Conversely, one can show that any unitary $\hat{U}$ which transforms $\hat{V}(z)$ as in Eq. (3) must correspond to an $S \in Sp(2n, \mathbb{R})$.

Weyl operators allow one to rewrite the canonical commutation relations as

$$\hat{V}(z)\hat{V}(z') = \exp[-\tfrac{i}{2} z^T \sigma_{2n} z'] \hat{V}(z + z') \,, \tag{4}$$

and permit a complete descriptions of the system in terms of (characteristic) complex functions. Specifically, any trace-class operator $\hat{\Theta}$ (in particular, any density operator) can be expressed as

$$\hat{\Theta} = \int \frac{d^{2n}z}{(2\pi)^n} \, \phi(\hat{\Theta}; z) \, \hat{V}(-z) \,, \tag{5}$$

where $d^{2n}z := dx_1 \cdots dx_n dy_1 \cdots dy_n$ and $\phi(\hat{\Theta}; z)$ is the characteristic function associated with the operator $\hat{\Theta}$ defined by

$$\phi(\hat{\Theta}; z) := \mathrm{Tr}[\, \hat{\Theta} \, \hat{V}(z) \,] \,. \tag{6}$$

Within this framework a density operator $\hat{\rho}$ of the n modes is said to represent a *Gaussian state* if its characteristic function $\phi(\hat{\rho}; z)$ has a Gaussian form, i.e.,

$$\phi(\hat{\rho}; z) = \exp[-\tfrac{1}{4} \, z^T \gamma z + i m^T z] \,, \tag{7}$$

with m being a real vector of mean values $m_j := \mathrm{Tr}[\, \hat{\rho} \, \hat{R}_j]$, and the $2n \times 2n$ real symmetric matrix γ being the *covariance matrix* [1, 4, 7] of $\hat{\rho}$. For generic density operators $\hat{\rho}$ (not only the Gaussian ones) the latter is defined as the variance of the canonical coordinates $\hat{R}$, i.e.,

$$\gamma_{j,j'} \;\; := \;\; \mathrm{Tr}\Big[\rho\big\{(R_j - m_j), (R_{j'} - m_{j'})\big\}\Big] \,, \tag{8}$$

with $\{\cdot, \cdot\}$ being the anti-commutator, and it is bound to satisfy the uncertainty relations

$$\gamma \geqslant i\sigma_{2n} \,, \tag{9}$$

with σ_{2n} being the commutation matrix (1). Up to an arbitrary vector m, the uncertainty inequality presented above uniquely characterizes the set of Gaussian states, i.e. any γ satisfying (9) defines a Gaussian state. Let us first notice that if γ satisfies (9) then it must be (strictly) positive definite $\gamma > 0$, and have $\mathrm{Det}[\gamma] \geqslant 1$. From Williamson theorem [23] it follows that there exists a symplectic $S \in Sp(2n, \mathbb{R})$ such that

$$\gamma = S \left[\begin{array}{cc} D & 0 \\ 0 & D \end{array} \right] S^T \,, \tag{10}$$

where $D := \mathrm{diag}(d_1, \cdots, d_n)$ is a diagonal matrix formed by the *symplectic eigenvalues* $d_j \geqslant 1$ of γ. For $S = \mathbb{1}_{2n}$ Eq. (10) gives the covariance matrix associated with thermal bosonic states. This also shows that any covariance matrix γ satisfying (9) can be written as

$$\gamma = SS^T + \Delta \,, \tag{11}$$

with $\Delta \geqslant 0$[1]. The extremal solutions of Eq. (11), i.e., $\gamma = SS^T$, are *minimal uncertainty solutions* and correspond to the *pure Gaussian states* of n modes (e.g., multi-mode squeezed vacuum states). They are uniquely determined by the condition $\mathrm{Det}[\gamma] = 1$ and satisfy the condition [18]

$$\gamma = -\sigma_{2n}(\gamma^{-1})\sigma_{2n} \,. \tag{13}$$

1.2 Bosonic Gaussian channels

In the Schrödinger picture evolution is described by applying the transformation to the states (i.e., the density operators), $\hat{\rho} \mapsto \Phi(\hat{\rho})$. In the Heisenberg picture the transformation is applied to the observables of the system, while leaving the states unchanged, $\hat{\Theta} \mapsto \Phi_H(\hat{\Theta})$. The two pictures are related through the identity $\mathrm{Tr}[\Phi(\hat{\rho})\hat{\Theta}] = \mathrm{Tr}[\hat{\rho}\Phi_H(\hat{\Theta})]$, which holds for all $\hat{\rho}$ and $\hat{\Theta}$. The map Φ_H is called the *dual* of Φ.

Due to the representation (5) and (6) any completely positive, trace preserving (CPT) transformation on the n-modes can be characterized by its action on the Weyl operators of the system in the Heisenberg picture (e.g., see Ref. [17]). In particular, a *bosonic Gaussian channel* (BGC) is defined as a map which, for all z, operates on $V(z)$ according to [3]

$$\hat{V}(z) \longmapsto \Phi_H(\hat{V}(z)) := \hat{V}(Xz)\, \exp[-\tfrac{1}{4} z^T Y z + i v^T z] \,, \tag{14}$$

with v being some fixed real vector of $\mathbb{R}^{2n}$, and with $Y, X \in \mathbb{R}^{2n\times 2n}$ being some fixed real $2n \times 2n$ matrices satisfying the complete positivity condition

$$Y \geqslant i\Sigma \qquad \text{with} \quad \Sigma := \sigma_{2n} - X^T \sigma_{2n} X \,. \tag{15}$$

In the context of BGCs the above inequality is the quantum channel counterpart of the uncertainty relation (9). Indeed up to a vector v, Eq. (15) uniquely determines the set of BGCs and bounds Y to be positive-semidefinite, $Y \geqslant 0$. However, differently from (9) in this case strict positivity is not a necessary prerequisite for Y. A completely positive map defined by Eqs. (14) and (15) will be referred to as bosonic Gaussian channel (BGC). As mentioned before, such a map is a model for a wide class of physical situations, ranging from communication channels such as optical fibers, to open quantum systems, and to dynamics in harmonic lattice systems. Whenever one has only partial access to the dynamics of a system that can be well-described

[1] This is indeed the matrix

$$\Delta := S \begin{bmatrix} D - \mathbb{1}_n & 0 \\ 0 & D - \mathbb{1}_n \end{bmatrix} S^T \tag{12}$$

with D as in Eq. (10) which is positive since $D \geqslant \mathbb{1}_n$.

by a time evolution governed by a Hamiltonian that is a quadratic polynomial in the canonical coordinates, one will arrive at a model described by Eqs. (14) and (15)[2].

An important subset of the BGCs is given by set of *Gaussian unitary* transformations which have $Y = 0$, $X \in Sp(2n, \mathbb{R})$, and v arbitrary. They include the canonical transformations of Eq. (3) (characterized by $v = 0$), and the displacement transformations (characterized by having $X = \mathbb{1}_{2n}$ and v arbitrary). The latter simply adds a phase to the Weyl operators and correspond to unitary transformations of the form $\Phi_H(\hat{V}(z)) := \hat{V}(-v)\hat{V}(z)\hat{V}(v) = \hat{V}(z)\, \exp[iv^T z]$.

In the Schrödinger picture the BGC transformation (14) induces a mapping of the characteristic functions of the form

$$\phi(\hat{\rho}; z) \longmapsto \phi(\Phi(\hat{\rho}); z) := \phi(\hat{\rho}; Xz)\, \exp[-\tfrac{1}{4} z^T Y z + iv^T z]\,, \tag{16}$$

which in turn yields the following transformation of the mean and the covariance matrix

$$\begin{aligned} m &\longmapsto m + v\,, \\ \gamma &\longmapsto X^T \gamma X + Y\,. \end{aligned} \tag{17}$$

Clearly, BGCs always map Gaussian input states into Gaussian output states.

For purposes of assessing quantum or classical information capacities, output entropies, or studying degradability or anti-degradability of a channel [16, 17, 14, 15], the full knowledge of the channel is not required: Transforming the input or the output with any unitary operation (say, Gaussian unitaries) will not alter any of these quantities. It is then convenient to take advantage of this freedom to simplify the description of the BGCs. To do so we first notice that the set of Gaussian maps is closed under composition. Consider then Φ' and Φ'' two BGCs described respectively by the elements X', Y', v' and X'', Y'', v''. The composition $\Phi'' \circ \Phi'$ where, in Schrödinger representation, we first operate with Φ' and then with Φ'', is still a BGC and it is characterized by the parameters

$$\begin{aligned} v &= (X'')^T v' + v''\,, \\ X &= X' X''\,, \\ Y &= (X'')^T\, Y'\, X'' + Y''\,. \end{aligned} \tag{18}$$

Exploiting these composition rules it is then easy to verify that the vector v can always be compensated by properly displacing either the input state or the output state (or both) of the channel. For instance by taking $X'' = \mathbb{1}_{2n}$, $Y'' = 0$ and $v'' = -v'$, Eq. (18) shows that Φ' is unitarily equivalent to the Gaussian channel Φ which has $v = 0$ and $X = X'$, $Y = Y'$. Therefore, without loss of generality, in the following we will focus on BGCs having $v = 0$.

More generally consider the case where we cascade a generic BGC Φ' described by matrices X', Y' as in Eq. (15) with a couple of canonical unitary transformation $\hat{U}_1$ and $\hat{U}_2$ described by the symplectic matrices S_1 and S_2 respectively. The resulting

[2] This set does not contain ideal Gaussian measurements [31], like optical homodyning 14.

BGC Φ is then described by the matrices

$$
\begin{aligned}
X &= S_1(X')S_2, \\
Y &= S_2^T(Y')S_2\,.
\end{aligned} \tag{19}
$$

For single mode ($n = 1$) this procedure induces a simplified canonical form [13, 17, 14] which, up to a Gaussian unitarily equivalence, allows one to focus only on transformations characterized by X and Y which, apart from some special cases, are proportional to the identity. In this paper we will generalize some of these results to an arbitrary number of modes n. To achieve this goal, in the following section we first present an explicit dilation representation in which the mapping (14) is described as a (canonical) unitary coupling between the n modes of the system and some extra *environmental* modes which are initially prepared into a Gaussian state. Then we will introduce the notion of minimal noise channel, showing a useful decomposition rule.

2 Unitary dilation theorem

In this section we introduce a general construction of unitary dilations of multi-mode quantum channels. Specifically we show that a CPT channel acting on n modes is a BGC if and only if it can be realized by invoking $\ell \leqslant 2n$ additional (environmental) modes E through the expression

$$
\Phi(\hat{\rho}) = \mathrm{Tr}_E[\hat{U}(\hat{\rho} \otimes \hat{\rho}_E)\hat{U}^\dagger]\,, \tag{20}
$$

where $\hat{\rho}$ is the input n-mode state of the system, $\hat{\rho}_E$ is a Gaussian state of an environment, $\hat{U}$ is a canonical unitary transformation which couples the system with the environment, and Tr_E denotes the partial trace over E. In case in which $\hat{\rho}_E$ is pure, Eq. (20) corresponds to a Stinespring dilation [24] of the channel Φ, otherwise it is a physical representation analogous to those employed in Refs. [16, 17] for the single mode case.

2.1 General dilations

In this subsection, we will construct Gaussian dilations, including a discussion of all rank-deficient cases, and will later focus on dilations involving the minimal number of modes. To proceed, we will first establish some conventions and notation. To start with, we write the commutation matrix of our $n + \ell$ modes in the block structure

$$
\sigma := \sigma_{2n} \oplus \sigma_{2\ell}^E = \begin{bmatrix} \sigma_{2n} & 0 \\ 0 & \sigma_{2\ell}^E \end{bmatrix} \begin{array}{l} \}\,2n \\ \}\,2\ell\,, \end{array} \tag{21}
$$

where σ_{2n} and $\sigma_{2\ell}^E$ are $2n \times 2n$ and $2\ell \times 2\ell$ commutation matrices associated with the system and environmental modes, respectively. For σ_{2n} we assume the structure as defined in Eq. (1). For $\sigma_{2\ell}^E$, in contrast, we do not make any assumption at this point, leaving open the possibility of defining it later on[3]. Accordingly, the canonical unitary transformation $\hat{U}$ of Eq. (20) will be uniquely determined by a $2(n+\ell) \times 2(n+\ell)$ real matrix $S \in Sp(2(n+\ell), \mathbb{R})$ of block form

$$S := \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} , \tag{22}$$

which satisfies the condition

$$S \sigma S^T = \sigma \,, \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{l} s_1 \, \sigma_{2n} \, s_1^T + s_2 \, \sigma_{2\ell}^E \, s_2^T = \sigma_{2n} \,, \\ \\ s_1 \, \sigma_{2n} \, s_3^T + s_2 \, \sigma_{2\ell}^E \, s_4^T = 0 \,, \\ \\ s_3 \, \sigma_{2n} \, s_3^T + s_4 \, \sigma_{2\ell}^E \, s_4^T = \sigma_{2\ell}^E \,. \end{array} \right. \tag{23}$$

In the above expressions, s_1 and s_4 are $2n \times 2n$ and $2\ell \times 2\ell$ real square matrices, while s_2 and s_3^T are $2n \times 2\ell$ real rectangular matrices. Introducing then the covariance matrices $\gamma \geqslant i\sigma_{2n}$ and $\gamma_E \geqslant i\sigma_{2\ell}^E$ of the states $\hat{\rho}$ and $\hat{\rho}_E$, the identity (20) can be written as

$$S \begin{bmatrix} \gamma & 0 \\ 0 & \gamma_E \end{bmatrix} S^T \Bigg|_{2n} = s_1 \, \gamma \, s_1^T + s_2 \, \gamma_E \, s_2^T = X^T \gamma X + Y, \tag{24}$$

where $|_{2n}$ denotes the upper principle submatrix of degree $2n$, and where $X, Y \in \mathbb{R}^{2n \times 2n}$ satisfying the condition (15) are the matrices associated with the channel Φ. In writing Eq. (24) we use the fact that due to the definition (21) the covariance matrix of the composite state $\hat{\rho} \otimes \hat{\rho}_E$ can be expressed as $\gamma \oplus \gamma_E$. With these definitions, the first part of the unitary dilation property (20) can be written as follows:

Proposition 1 (Unitary dilations of Gaussian channels) *Let γ_E be the covariance matrix of a Gaussian state of ℓ modes and let $S \in Sp(2(n+\ell), \mathbb{R})$ be a symplectic transformation. Then there exists a symmetric $2n \times 2n$-matrix $Y \geqslant 0$ and a $2n \times 2n$-matrix X satisfying the condition (15), such that Eq. (24) holds for all γ.*

Proof: The proof is straightforward: We write S in the block form (22) and take $X = s_1^T$ and $Y = s_2 \gamma_E s_2^T$. Since γ_E is a covariance matrix of ℓ modes, $\gamma_E - i\sigma_{2\ell} \geqslant 0$ and therefore $s_2(\gamma_E - i\sigma_\ell)s_2^T \geqslant 0$. This leads to Eq. (15) through the identity the symplectic condition $s_1 \sigma_{2n} s_1^T + s_2 \sigma_{2\ell} s_2^T = \sigma_{2n}$ which follows by comparing the upper principle submatrices of degree n of both terms of Eq. (23). ■

This proves that any CPT map obtained by coupling the n modes with a Gaussian state of ℓ environmental bosonic modes through a Gaussian unitary $\hat{U}$ is a BGC. The converse property is more demanding. In order to present it we find it useful to state first the following

[3] With this choice the canonical commutation relations of the $n+\ell$ mode read as $[\hat{R}_j, \hat{R}_{j'}] = i\sigma_{j,j'}$ where $\hat{R} := (\hat{Q}_1, \cdots, \hat{Q}_n; \hat{P}_1, \cdots, \hat{P}_n; \hat{r}_1, \cdots, \hat{r}_{2\ell})$ with $\hat{Q}_j, \hat{P}_j$ being the canonical coordinates of the j-th system mode and with and $\hat{r}_1, \cdots, \hat{r}_{2\ell}$ being some ordering of the canonical coordinates $\hat{Q}_1^E, \hat{P}_1^E; \cdots; \hat{Q}_\ell^E, \hat{P}_\ell^E$ of the environmental modes. For instance, taking $\sigma_{2\ell}^E = \sigma_{2\ell}$ corresponds to have $\hat{R} := (\hat{Q}_1, \cdots, \hat{Q}_n; \hat{P}_1, \cdots, \hat{P}_n; \hat{Q}_1^E, \cdots, \hat{Q}_\ell^E; \hat{P}_1^E, \cdots, \hat{P}_\ell^E)$.

Lemma 1 (Extensions of symplectic forms) *Let, for some skew symmetric $\sigma_{2\ell}^E$, s_1 and s_2 be $2n \times 2n$ and $2n \times 2\ell$ real matrices forming a symplectic system, i.e., $s_1\,\sigma_{2n}\,s_1^T + s_2\,\sigma_{2\ell}^E\,s_2^T = \sigma_{2n}$. Then we can always find real matrices s_3 and s_4 such that S of Eq. (22) is symplectic with respect to the commutation matrix (21).*

Proof: Since the rows of S form a symplectic basis, given s_1 and s_2 (an incomplete symplectic basis), it is always possible to find s_3 and s_4 as above. The proof easily follows from a skew-symmetric version of the Gram-Schmidt process to construct a symplectic basis [25]. For a special subset of BGCs, in Sec. 2.5 we will present an explicit expression for S based on a simplified (canonical) representation of the X matrix that defines Φ. See also Appendix A. ■

Due to the above result, the possibility of realizing unitary dilation Eq. (20) for a generic BGC described by the matrices X and $Y \geqslant i\Sigma = i(\sigma_{2n} - X^T\sigma_{2n}X)$, can be proven by simply taking $s_1 = X^T$ and finding some $2n \times 2\ell$ real matrix s_2 and an ℓ-mode covariance matrix $\gamma_E \geqslant i\sigma_{2\ell}^E$ that solve the equations

$$s_2\,\sigma_{2\ell}^E\,s_2^T = \sigma_{2n} - s_1\,\sigma_{2n}\,s_1^T = \Sigma\,, \tag{25}$$

$$s_2\,\gamma_E\,s_2^T = Y\,. \tag{26}$$

With this choice in fact Eq. (24) is trivially satisfied for all γ, while s_1 and s_2 can be completed to a symplectic matrix $S \in Sp(2(n+\ell),\mathbb{R})$. Note that $Sp(2(n+\ell),\mathbb{R})$ stands for the standard symplectic group here. The unitary dilation property (20) can hence be restated as follows:

Theorem 1 (Unitary dilations of Gaussian channels: Converse implication) *For any real $2n\times 2n$-matrices X and Y satisfying the condition (15), there exist ℓ smaller than or equal to $2n$, $S \in Sp(2(n+\ell),\mathbb{R})$, and a covariance matrix γ_E of ℓ modes, such that Eq. (24) is satisfied.*

Proof: As already noticed the whole problem can be solved by assuming $s_1 = X^T$ and finding s_2 and γ_E that satisfy Eqs. (25) and (26). We start by observing that the $2n \times 2n$ matrix Σ defined in Eq. (15) is skew-symmetric, i.e., $\Sigma = -\Sigma^T$. Moreover according to Eq. (15) its support must be contained in the support of Y, i.e., Supp$[\Sigma] \subseteq$ Supp$[Y]$. Consequently given $k :=$ rank$[Y]$ and $r :=$ rank$[\Sigma]$ as the ranks of Y and Σ, respectively, one has that $k \geqslant r$. We can hence identify three different regimes:

(i) $k = 2n$, $r = 2n$, i.e., both Y and Σ are full rank and hence invertible. Loosely speaking, this means that all the noise components in the channel are basically quantum (although may involve classical noise as well).

(ii) $k = 2n$ and $r < 2n$, i.e., Y is full rank and hence invertible, while Σ is singular. This means that the some of the noise components can be purely classical, but still nondegenerate.

(iii) $2n > k \geqslant r$, i.e., both Y and Σ are singular. There are noise components with zero variance.

Even though (i) and (ii) admit similar solutions, it is instructive to analyze them separately. In the former case in fact there is a simple direct way of constructing a physical dilation of the channel with $\ell = n$ environmental modes.

(i) Since Σ is skew-symmetric and invertible there exists an $O \in O(2n, \mathbb{R})$ orthogonal such that

$$O\Sigma O^T \;=\; \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}, \tag{27}$$

where $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_n)$ and $\mu_i > 0$ for all $i = 1, \cdots, n$ (see page 107 in Ref. [26]). Hence $K := M^{-1/2}O$ with $M := \mu \oplus \mu$ satisfies

$$K\Sigma K^T = \sigma_{2n}\,. \tag{28}$$

Taking then $s_2 := K^{-1}$ we get[4]

$$s_2\, \sigma_{2n}\, s_2^T = K^{-1}\, \sigma_{2n}\, K^{-T} = \Sigma\,, \tag{29}$$

which corresponds to Eq. (25) for $\ell = n$. Since $s_1 = X^T$, Lemma 1 guarantees that this is sufficient to prove the existence of S. The condition (24) finally follows by taking $\gamma_E = KYK^T$ which is strictly positive (indeed K is invertible and $Y > 0$ because it has full rank) and which satisfies the uncertainty relation (9), i.e.,

$$Y \geqslant i\Sigma \quad \Longrightarrow \quad \gamma_E = KYK^T \geqslant iK\Sigma K^T = i\sigma_{2n}\,. \tag{30}$$

This shows that the channel admits a unitary dilation of the form as specified in Eq. (20) with $\ell = n$ environmental modes with commutation matrix, $\sigma_{2n}^E = \sigma_{2n}$ – see discussion after Eq. (21). Such a solution, however, will involve a pure state $\hat{\rho}_E$ only if $\mathrm{Det}[\gamma_E] = 1$, i.e.,

$$\mathrm{Det}[Y]\mathrm{Det}[K]^2 = 1 \quad \Longleftrightarrow \quad \mathrm{Det}[Y] = \mathrm{Det}[\Sigma]\,. \tag{31}$$

When $\mathrm{Det}[\gamma_E] > 1$, i.e., $\mathrm{Det}[Y] > \mathrm{Det}[\Sigma]$, we can still construct a pure dilation by simply adding further n modes which purify the state associated with the covariance matrix γ_E and by extending the unitary operator $\hat{U}$ associated with S as the identity operator on them. For details see the discussion of case (ii) given below. This corresponds to constructing a unitary dilation (20) with the pure state $\hat{\rho}_E$ being defined on $\ell = 2n$ modes.

(ii) In this case Y is still invertible, while Σ is not. Differently from the approach we adopted in solving case (i), we here derive directly a Stinespring unitary dilation, i.e., we construct a solution with a pure γ_E that involves $\ell = 2n$ environmental modes. In the next section, however, we will show that, dropping the purity requirement, one can construct unitary dilation that involves $\hat{\rho}_E$ with only $\ell = 2n - r/2$ modes.

[4] From now on, the symbol A^{-T} will be used to indicate the transpose of the inverse of the matrix A, i.e., $A^{-T} := (A^{-1})^T = (A^T)^{-1}$.

To find s_2 and γ_E which solve Eqs. (25) and (26), it is useful to first transform Y into a simpler form by a congruent transformation, i.e.,

$$CYC^T = \mathbb{1}_{2n}\,, \tag{32}$$

with $C \in Gl(2n, \mathbb{R})$ being not singular, e.g., $C := Y^{-1/2}$. From Eq. (15) it then follows that

$$\mathbb{1}_{2n} \geqslant i\Sigma'\,, \tag{33}$$

with $\Sigma' := Y^{-1/2}\,\Sigma\, Y^{-1/2}$ being skew-symmetric (i.e., $\Sigma' = -(\Sigma')^T$) and singular with rank$[\Sigma']$ = rank$[\Sigma] = r$ [26]. We then observe that introducing

$$s_2 = Y^{1/2}\, s_2'\,, \tag{34}$$

the conditions (25) and (26) can be written as

$$\begin{aligned} s_2'\,\sigma_{2\ell}^E\,(s_2')^T &= \Sigma'\,, & (35)\\ s_2'\,\gamma_E\,(s_2')^T &= \mathbb{1}_{2n}\,. & (36)\end{aligned}$$

Finding s_2' and γ_E which satisfy these expressions will provide us also a solution of Eqs. (25) and (26).

As in the case of Eq. (27), there exists an orthogonal matrix $O \in O(2n, \mathbb{R})$ which transforms the skew-symmetric matrix Σ' in a simplified block form. In this case however, since Σ' is singular, we find [26]

$$O\Sigma' O^T = \begin{array}{c} \left[\begin{array}{cc|cc} 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{37}$$

where now $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_{r/2})$ is the $r/2 \times r/2$ diagonal matrix formed by the strictly positive eigenvalues of $|\Sigma'|$ which satisfy the conditions $1 \geqslant \mu_j > 0$, this being equivalent with

$$\mathbb{1}_{r/2} \geqslant \mu, \tag{38}$$

as a consequence of inequality (33). Define then $K := M^{-1/2}\, O$ with

$$M = \left[\begin{array}{cc|cc} \mu & 0 & 0 & 0 \\ 0 & \mathbb{1}_{n-r/2} & 0 & 0 \\ \hline 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mathbb{1}_{n-r/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2. \end{array} \tag{39}$$

It satisfies the identity

$$K\Sigma' K^T \;=\; \left[\begin{array}{cc|cc} 0 & 0 & \mathbb{1}_{r/2} & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\mathbb{1}_{r/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2. \end{array} \tag{40}$$

To show that Eqs. (35) and (36) admit a solution we take $\ell = 2n$ and write $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n} = \sigma_{4n}$ with σ_{2n} as in Eq. (1). With these definitions the $2n \times 4n$ rectangular matrix s_2' can be chosen to have the block structure

$$s_2' \;=\; \left[\, K^{-1} \,\middle|\, O^T A \,\right] , \tag{41}$$

with A being the following $2n \times 2n$ symmetric matrix

$$A = A^T \;=\; \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & \mathbb{1}_{n-r/2} \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \mathbb{1}_{n-r/2} & & \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2. \end{array} \tag{42}$$

By direct substitution one can easily verify that Eq. (35) is indeed satisfied, see Appendix B for details. Inserting Eq. (41) into Eq. (36) yields now the following equation

$$\alpha + A\,\delta^T + \delta\,A^T + A\,\beta\,A^T = M^{-1} , \tag{43}$$

for the $4n \times 4n$ covariance matrix

$$\gamma_E = \left[\begin{array}{cc} \alpha & \delta \\ \delta^T & \beta \end{array}\right] , \tag{44}$$

see Appendix C for details. A solution can be easily derived by taking

$$\alpha = \beta = \left[\begin{array}{cc|cc} \mu^{-1} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \xi\mathbb{1}_{n-r/2} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \mu^{-1} & 0 \\ & & 0 & \xi\mathbb{1}_{n-r/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{45}$$

with $\xi = 5/4$ and

$$\delta = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & f(\mu^{-1}) & 0 \\ & & 0 & f(\xi\mathbb{1}_{n-r/2}) \\ \hline f(\mu^{-1}) & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & f(\xi\mathbb{1}_{n-r/2}) & & \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{46}$$

with $f(\theta) := -(\theta^2 - \mathbb{1})^{1/2}$. For all diagonal matrices μ compatible with the constraint (38) the resulting γ_E satisfies the uncertainty relation $\gamma_E \geqslant i\sigma_{4n}$. Moreover since it has Det$[\gamma_E] = 1$, this is also a minimal uncertainty state, i.e., a pure Gaussian state of $2n$ modes. It is worth stressing that for $r = 2n$, i.e., when also the rank of Σ is maximum, the above solution provides an alternative derivation of the unitary dilation discussed in the part (i) of the theorem. In this case the covariance matrix γ_E has block elements

$$\alpha = \beta = \left[\begin{array}{cc} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \end{array} , \qquad \delta = \left[\begin{array}{cc} 0 & f(\mu^{-1}) \\ f(\mu^{-1}) & 0 \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \end{array} , \tag{47}$$

where μ is now a strictly positive $n \times n$ matrix, while Eqs. (34) and (41) yield

$$s_2 := Y^{1/2} O^T \left[\begin{array}{c|c|c|c} \mu^{1/2} & 0 & 0 & 0 \\ \hline 0 & \mu^{1/2} & 0 & 0 \end{array} \right] \begin{array}{l} \} \, n \\ \} \, n. \end{array} \tag{48}$$

(iii) Here both Y and Σ are singular. This case is very similar to case (ii). Here, the dilation can be constructed by introducing a strictly positive matrix $\bar{Y} > 0$ which satisfies the condition

$$\Pi \, \bar{Y} \, \Pi = Y \,, \tag{49}$$

with Π being the projector onto the support of Y. Such a $\bar{Y}$ always exists ($\bar{Y} = Y + (\mathbb{1} - \Pi)$). By construction, it satisfies the inequality $\bar{Y} \geqslant Y \geqslant i\Sigma$. According to Sec. 1.2, $\bar{Y}$ and X define thus a BGC. Moreover, since $\bar{Y}$ is strictly positive, it has full rank. Therefore, we can use part (ii) of the proof to find a $2n \times 2\ell$ matrix $\bar{s}_2$ and $\bar{\gamma}_E \geqslant i\sigma_{2\ell}$ which satisfy the conditions (25) and (26), i.e.

$$\begin{aligned} \bar{s}_2 \, \sigma_{2\ell}^E \, \bar{s}_2^T &= \Sigma \,, & (50) \\ \bar{s}_2 \, \bar{\gamma}_E \, \bar{s}_2^T &= \bar{Y} \,. & (51) \end{aligned}$$

A unitary dilation for the channel Y, X is then obtained by choosing $\gamma_E = \bar{\gamma}_E$ and $s_2 = \Pi \bar{s}_2$. In fact from Eq. (51) we get

$$s_2 \, \gamma_E \, s_2^T = \Pi \, \bar{s}_2 \, \bar{\gamma}_E \, \bar{s}_2^T \, \Pi = \Pi \, \bar{Y} \, \Pi = Y \,, \tag{52}$$

while from Eq. (50)

$$s_2 \, \sigma_{2\ell}^E \, s_2^T = \Pi \, \bar{s}_2 \, \sigma_{2\ell}^E \, \bar{s}_2^T \, \Pi = \Pi \, \Sigma \, \Pi = \Sigma \,, \tag{53}$$

where we have used the fact that Supp$[\Sigma] \subseteq$ Supp$[Y]$. ■

In proving the second part of the unitary dilations theorem we provided explicit expressions for the environmental state $\hat{\rho}_E$ of Eq. (20). Specifically such a state is given by the pure $2n$ mode Gaussian state $\hat{\rho}_E$ characterized by the covariance matrix γ_E of elements (45) and (46). A trivial observation is that this can always be replaced by the $2n$ modes vacuum state $|\emptyset\rangle\langle\emptyset|$ having the covariance matrix $\gamma_E^{(0)} = \mathbb{1}_{2n}$. This is a consequence of the obvious property that according to Eq. (11) all pure Gaussian states are equivalent to $|\emptyset\rangle\langle\emptyset|$ up to a Gaussian unitary transformation. On the level of covariance matrices, Gaussian unitaries correspond to symplectic transformations. For a remark on unitarily equivalent dilations, see also Appendix D. Hence, by means of a congruence with an appropriate symplectic transformation, we immediately arrive at the following corollary:

Corollary 1 (Gaussian channels with pure Gaussian dilations) *Any n-mode Gaussian channel Φ admits a Gaussian unitary dilation (20) with $\hat{\rho}_E = |\emptyset\rangle\langle\emptyset|$ being the vacuum state on $2n$ modes.*

2.2 Reducing the number of environmental modes

An interesting question is the characterization of the minimal number of environmental modes ℓ that need to be involved in the unitary dilation (20). From Theorem 1 we know that such number is certainly smaller than or equal to twice the number n of modes on which the BGC is operating: We have in fact explicitly constructed one of such representations that involves $\ell = 2n$ modes in a minimal uncertainty, i.e., pure Gaussian state. We also know, however, that there are situations[5] in which ℓ can be reduced to just n: This happens for instance for BGCs Φ with rank$[Y] =$ rank$[\Sigma] = 2n$, i.e., case (i) of Theorem 1. In this case one can represent the channel Φ in terms of a Gaussian unitary coupling with $\ell = n$ environmental modes which are prepared into a Gaussian state with covariance matrix

$$\gamma_E = KYK^T \,, \tag{54}$$

– see Eqs. (30). In general, this will not be of Stinespring form (not be a pure unitary dilation) since γ_E is not a minimal uncertainty covariance matrix. In fact, for $n = 1$ this corresponds to the physical representation of Φ of Refs. [17]. However if Y and X satisfy the condition (31), our analysis provides a unitary dilation involving merely $\ell = n$ modes in a pure Gaussian state.

We can then formulate a necessary and sufficient condition for the channels Φ of class (i) which can be described in terms of n environmental modes prepared into a pure state. It is given by

$$Y = \Sigma\, Y^{-1}\, \Sigma^T \,, \tag{55}$$

which follows by imposing the minimal uncertainty condition (13) to the n-mode covariance matrix (54) and by using (28). Similarly one can verify that given a pure n-modes Gaussian state $\hat{\rho}_E$ and an $S \in Sp(4n, \mathbb{R})$ (22) with an invertible subblock s_2, then the corresponding BGC satisfies condition (55). The above result can be strengthened by looking at the solutions for channels of class (ii) of which the channel of class (i) are a proper subset.

To achieve this goal, let us first note that with the choice we made on $\sigma_{2\ell}^E = \sigma_{4n}$, the two matrices α and β of Eq. (45) are $2n \times 2n$ covariance matrices for two sets of independent n bosonic modes satisfying the uncertainty relations (9) with respect to the form σ_{2n}. In turn, the matrices δ and δ^T of Eq. (46) represent cross-correlation terms among such sets. After all, the entire covariance matrix γ_E corresponds to a pure Gaussian state.

They key point is now the observation that in Eq. (43), the matrix A couples only with those rows and columns of the matrices δ and β which contain elements $\xi \mathbb{1}_{n-r/2}$ or $f(\xi \mathbb{1}_{n-r/2})$: As far as A is concerned, one could indeed replace the element μ^{-1} and $f(\mu^{-1})$ of such matrices with zeros. The only reason we keep these element the way they are in Eqs. (45) and (46), is to render γ_E the covariance matrix of a minimal uncertainty state. In other words, the elements of δ and β proportional to μ^{-1} or $f(\mu^{-1})$ are only introduced to purify the corresponding element of the submatrix α, which is in itself hence a covariance matrix of a mixed Gaussian state.

[5]Not mentioning the trivial case of Gaussian unitary transformation which does not require any environmental mode to construct a unitary dilation.

Suppose then that μ of Eq. (38) has (say) the first $r'/2$ eigenvalues equal to 1, i.e., $\mu_1 = \mu_2 = \cdots = \mu_{r'/2} = 1$ while for $j \in \{r'/2+1, \cdots, r/2\}$ we have that $\mu_j \in (0,1)$. In this case the corresponding sub-matrix of α associated with those elements represent a pure Gaussian state, specifically the vacuum state. Accordingly, there is no need to add further modes to purify them. Taking this into account, one can hence reduce the number of environmental modes ℓ_{pure} that allows one to represent Φ as in Eq. (20) in term of a *pure state* $\hat{\rho}_E$ from $2n$ to

$$\ell_{\text{pure}} = n + (n - r'/2) = 2n - r'/2 \,, \tag{56}$$

i.e. we need the n modes of α plus $n - r'/2$ additional modes of β to purify those of α which are not in a pure state already. An easy way to characterize the parameter r' is to observe that, according to Eq. (37), it corresponds to the number of eigenvalues having modulus 1 of the matrix of $O\Sigma' O^T$, i.e.,

$$\begin{aligned} r' &= 2n - \text{rank}[\mathbb{1}_{2n} - O\Sigma'(\Sigma')^T O^T] = 2n - \text{rank}[\mathbb{1}_{2n} - \Sigma'(\Sigma')^T] \\ &= 2n - \text{rank}[Y - \Sigma\, Y^{-1}\, \Sigma^T]\,. \end{aligned} \tag{57}$$

The explicit expressions for corresponding values of γ_E and s_2 are given in Appendix C.1. Here we notice that for $r' = r = 2n$ we get $\ell_{\text{pure}} = n$. This should correspond to the channels (55) of class (i) for which one can construct a unitary dilation with pure input states. Indeed, according to Eq. (57), when $r' = 2n$ the matrix $Y - \Sigma\, Y^{-1}\, \Sigma^T$ must be zero, leading to the identity (55).

Taking into account that $r' \leqslant r = \text{rank}[\Sigma]$, a further reduction in the number of modes ℓ can be obtained by dropping the requirement of γ_E being a minimal uncertainty covariance matrix. Indeed, an alternative unitary representation (20) of Φ can be constructed with only

$$\ell = n + (n - r/2) = 2n - r/2\,, \tag{58}$$

environmental modes (see Appendix C.2 for the explicit solution).

The whole analysis can be finally generalized to the BGCs of class (iii), corresponding to channels that have non invertible matrices Y. We have seen in fact that, in this case, the state $\hat{\rho}_E$ which provides us the unitary dilation of Theorem 1 is constructed by replacing Y with the strictly positive operator $\bar{Y}$ of Eq. (49). Therefore for these channels ℓ_{pure} of Eq. (56) is defined by Eq. (57) with Y replaced by $\bar{Y}$, i.e.

$$r' = 2n - \text{rank}[\bar{Y} - \Sigma\, \bar{Y}^{-1}\, \Sigma^T]\,. \tag{59}$$

Taking $\bar{Y} := Y + (\mathbb{1}_{2n} - \Pi)$ with Π being the projector on Supp$[Y]$ this gives,

$$\begin{aligned} r' &= 2n - \text{rank}[\bar{Y} - \Sigma\, Y^{\ominus 1}\, \Sigma^T] = 2n - \text{rank}[Y - \Sigma\, Y^{\ominus 1}\, \Sigma^T] - \text{rank}[\mathbb{1}_{2n} - \Pi] \\ &= k - \text{rank}[Y - \Sigma\, Y^{\ominus 1}\, \Sigma^T]\,, \end{aligned} \tag{60}$$

where $k = \text{rank}[Y] = \text{rank}[\Pi]$, where $Y^{\ominus 1} := \Pi \bar{Y}^{-1} \Pi$ denotes the Moore-Penrose inverse [26] of Y, and where we have used the fact that Supp$[\Sigma] \subseteq$ Supp$[Y]$. Remembering then that for channels of class (ii) $k = 2n$ and $Y^{\ominus 1} = Y^{-1}$ these results can be summarized as follows:

Theorem 2 (Dilations of BGCs involving fewer additional modes) *Given Φ a BGC described by matrices X and Y satisfying the conditions (15) and characterized by the quantities*

$$r = \mathit{rank}[\Sigma]\,, \qquad r' = \mathit{rank}[Y] - \mathit{rank}[Y - \Sigma Y^{\ominus 1}\Sigma^T]\,. \tag{61}$$

Then it is possible to construct a unitary dilation (20) of Stinespring form (i.e., involving a pure Gaussian state $\hat{\rho}_E$) with at most $\ell_{pure} = 2n - r'/2$ environmental modes. It is also always possible to construct a unitary dilation (20) using $\ell = 2n - r/2$ environmental modes which are prepared in a Gaussian, but not necessarily pure state.

It is worth stressing that, for channel of class (ii) and (iii), the Theorem 2 only provides upper bounds for the minimal values of ℓ and ℓ_{pure}. Only in the generic case (i) these bounds coincide with the real minima.

2.3 Minimal noise channels

In a very analogous fashion to the extremal covariance matrices corresponding to pure Gaussian states, one can introduce the concept of a minimal noise channel. In this section we review the concept of such minimal noise channels [18] and provide criteria to characterize them. Given $X, Y \in \mathbb{R}^{2n\times 2n}$ satisfying the inequality (15), any other $Y' = Y + \Delta Y$ with $\Delta Y \geqslant 0$ will also satisfy such condition, i.e.,

$$Y' \geqslant Y \geqslant i(\sigma_{2n} - X^T \sigma_{2n} X)\,. \tag{62}$$

Furthermore, due to the compositions rules (18), the BGC Φ' associated with the matrices X, Y' can be described as the composition

$$\Phi' = \Psi \circ \Phi\,, \tag{63}$$

between the channel Φ associated with the matrices X, Y, and the channel Ψ described by the matrices $X = \mathbb{1}_n$ and $Y = \Delta Y$. The latter belongs to a special case of BGC that includes the so called *additive classical noise channels* [17, 3, 4] – see Sec. 2.4 for details.

For any $X \in \mathbb{R}^{2n\times 2n}$, one can then ask how much *noise* Y it is necessary to add in order to obtain a map satisfying the condition (15). This gives rise to the notion of *minimal noise* [18], as the extremal solutions Y of Eq. (15) for a given X. The corresponding *minimal noise channels* are the natural analogue of the Gaussian pure state and allows one to represent any other BGC as in Eq. (63) with a proper choice of the additive classical noise map Ψ.

Let us start considering the case of a generic channel Φ' of class (i) described by matrices X and Y'. According to Theorem 1 it admits unitary dilation with $\ell = n$ modes described by some covariance matrix γ'_E satisfying the condition

$$Y' = s_2\, \gamma'_E\, s_2^T\,, \tag{64}$$

for some proper $2n \times 2n$ real matrix s_2. According to Eq. (11) γ_E can be written as

$$\gamma'_E = \gamma_E + \Delta\,, \tag{65}$$

with $\Delta \geqslant 0$ and γ_E minimal uncertainty state. Therefore writing $Y = s_2 \gamma_E s_2^T$ and $\Delta Y = s_2 \Delta s_2^T$ we can express Φ' as in (63), where now Φ is the BGC associated with the minimal noise environmental state γ_E. Most importantly since the decomposition (65) is optimal for γ'_E, the channel Φ is an extremal solution of Eq. (15). We stress that by construction Φ is still a channel of class (i): in fact it has the same Σ as Φ', while Y is still strictly positive since $\gamma_E > 0$ and s_2 is invertible – see Eq. (64). We can then use the results of Sec. 2.2 to claim that Φ must satisfy the equality (55). This leads us to establish three equivalent necessary and sufficient conditions for minimal noise channels of class (i):

$$(m_1) \qquad Y = \Sigma Y^{-1} \Sigma^T \,, \tag{66}$$

$$(m_2) \qquad \text{Det}[Y] = \text{Det}[\Sigma] \,, \tag{67}$$

$$(m_3) \qquad r = r' \,, \tag{68}$$

with r and r' as in Eq. (61). Since for class (i) we have that $r = 2n$, the minimal noise condition m_3 simply requires the eigenvalues of the matrix μ of Eq. (37) to be equal to unity. Similarly, minimal noise channels in case (ii) and (iii) can be characterized.

Theorem 3 (Minimal noise condition) *A Gaussian bosonic channel characterized by the matrices Y and $X \in \mathbb{R}^{2n \times 2n}$ is a minimal noise channel if and only if*

$$Y = \Sigma Y^{\ominus 1} \Sigma^T, \tag{69}$$

where, as throughout this work, $\Sigma = \sigma_{2n} - X^T \sigma_{2n} X$.

Proof: The complete positivity condition (15) of a generic BGC is a positive semi-definite constraint for the symplectic form Σ, corresponding to the constraint $\gamma - i\sigma_{2n} \geqslant 0$ in case of covariance matrices of states of n modes. In general, $r = \text{rank}[\Sigma]$ is not maximal, i.e., not equal to $2n$. When identifying the minimal solutions of the inequality (15), without loss of generality we can look for the minimal solutions of

$$Y' - i\Sigma' \geqslant 0, \tag{70}$$

where here

$$\Sigma' = \left[\begin{array}{cc|c} \begin{array}{c|c} 0 & \mu \\ \hline -\mu & 0 \end{array} & & \\ \hline & & 0 \end{array} \right], \tag{71}$$

with $\mu > 0$ being diagonal of rank $r/2$ (here $Y' = OYO^T$ and $\Sigma' = O\Sigma O^T$ with $O \in O(2n, \mathbb{R})$ orthogonal). The minimal solutions of inequality (70) are then given by $Y' = SS^T \oplus 0$, where S is a $r \times r$ matrix satisfying

$$S \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} S^T = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}, \tag{72}$$

so a symplectic matrix with respect to the modified symplectic form, so an element of $\{M \in Gl(r, \mathbb{R}) : M = (\mu^{1/2} \oplus \mu^{1/2}) S (\mu^{-1/2} \oplus \mu^{-1/2}), S \in Sp(r, \mathbb{R})\}$. From this, it follows that the minimal solutions of (70) are exactly given by the solutions of $Y' = \Sigma' (Y')^{\ominus 1} (\Sigma')^T$, from which the statement of the theorem follows. ■

2.4 Additive classical noise channel

In this subsection we focus on the maps Ψ which enter in the decomposition (63). They are characterized by having $X = \mathbb{1}_{2n}$ and $Y \geqslant 0$. Note that with this choice the condition (15) is trivially satisfied. This is the classical noise channel that has frequently been considered in the literature (for a review, see, e.g., Ref. [4]). For completeness of the presentation, we briefly discuss this class of multi-mode BGC.

If the matrix Y is strictly positive, the channel Ψ is the multi-mode generalization of the single mode additive classical noise channel [17, 3, 4]. In the language of Ref. [17], these are the maps which have a canonical form B_2 according to [17]). Indeed, one can show that these maps are the (Gaussian) unitary equivalent to a collection of n single mode additive classical noise maps. To see this, let us apply symplectic transformations (S_1 and S_2) before and after the channel Ψ. Following Eq. (19) this leads to $\{\mathbb{1}_n, Y\} \mapsto \{S_1 S_2, \quad S_2^T Y S_2\}$. Now, since $Y > 0$, according to *Williamson's theorem* [23], we can find a $S_2 \in Sp(2n, \mathbb{R})$ such that $S_2^T Y S_2$ is diagonal diag$(\lambda_1, \cdots, \lambda_n, \lambda_1, \cdots, \lambda_n)$ with $\lambda_i > 0$. We can then take $S_1 = S_2^{-1}$ to have $S_1 S_2 = \mathbb{1}_{2n}$. For $Y \geqslant 0$ but not $Y > 0$, the maps Ψ that enter the decomposition Eq. (63) however include also channels which are not unitarily equivalent to a collection of B_2 maps. An explicit example of this situation is constructed in Appendix E.

2.5 Canonical form for generic channels

Analogously to Refs. [17, 13, 14], any BGC Φ described by the transformation Eq. (17) can be simplified through unitarily equivalence by applying unitary canonical transformations before and after the action of the channel which induces transformations of the form (19). Specifically, given a n-mode Gaussian channel Φ described by matrix X and Y we can transform it into a new n-mode Gaussian channel Φ_c described by the matrices

$$X_c = S_1 X S_2 \,, \qquad Y_c = S_2^T Y S_2 \,, \tag{73}$$

with $S_{1,2} \in Sp(2n, \mathbb{R})$. As already discussed in the introductory sections, from an information theoretical perspective Φ and Φ_c are equivalent in the sense that, for instance, their unconstrained quantum capacities coincide. We can then simplify the analysis of the n-mode Gaussian channels by properly choosing S_1 and S_2 to induce a parametrization of the interaction part (i.e., X) of the evolution. The resulting canonical form follows from the generalization of the Williamson theorem [23] presented in Ref. [20]. According to this result, for every non-singular matrix $X \in Gl(2n, \mathbb{R})$, there exist matrices $S_{1,2} \in Sp(2n, \mathbb{R})$ such that

$$X_c = S_1 X S_2 = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix} , \tag{74}$$

where J^T is a $n \times n$ block-diagonal matrix in the real Jordan form [26]. This can be developed a little further by constructing a canonical decomposition for the symplectic matrix S associated with the unitary dilation (20) of the channel.

For the sake of simplicity in the following we will focus on the case of generic quantum channels Φ which have non-singular $X \in Gl(2n, \mathbb{R})$ and belong to the class (i) of Theorem 1 (i.e., which have $r = \text{rank}[\Sigma] = 2n$). Under these conditions X must admit a canonical decomposition of the form (74) in which all the eigenvalues of J are different from 1. In fact one has

$$\Sigma = \sigma_{2n} - X^T \sigma_{2n} X = S_2^{-T} \left[\sigma_{2n} - X_c^T \sigma_{2n} X_c\right] S_2^{-1} = S_2^{-T} \, \Sigma_c \, S_2^{-1} \,, \tag{75}$$

with Σ_c being the skew-symmetric matrix associated with the channel Φ_c, i.e.,

$$\Sigma_c := \left[\begin{array}{cc} 0 & \mathbb{1}_n - J \\ J^T - \mathbb{1}_n & 0 \end{array} \right] . \tag{76}$$

Since $\text{rank}[\Sigma_c] = \text{rank}[\Sigma] = 2n$, it follows that J cannot have eigenvalues equal to 1. Similarly, it is not difficult to see that if X has a canonical form (74) with all the eigenvalues of J being different from 1, then Φ and Φ_c are of class (i). However, a special case in which $X = \mathbb{1}_{2n}$ is investigated in details in Appendix E.

Consider then a unitary dilation (20) of the channel Φ_c constructed with a not necessarily pure Gaussian state $\hat{\rho}_E$ of $\ell = n$ environmental modes. According to the above considerations, such a dilation always exists. Let $S \in Sp(4n, \mathbb{R})$ be the $4n \times 4n$ real symplectic transformation (22) associated with the corresponding unitary $\hat{U}$. Assuming $s_1 = X_c^T$, an explicit expression for this dilation can be obtained by writing

$$s_4 = \left[\begin{array}{cc} \mathbb{1}_n & 0 \\ 0 & J' \end{array} \right], \; s_j = \left[\begin{array}{cc} F_j & 0 \\ 0 & G_j \end{array} \right], \tag{77}$$

where, for $j = 2, 3$, F_j, G_j are $n \times n$ real matrices. Imposing Eqs. (23), one obtains the following relations

$$\begin{array}{ll} J^T + F_2 G_2^T = \mathbb{1}_n \,, & J'^T + F_3 G_3^T = \mathbb{1}_n \,, \\ G_3^T + F_2 J'^T = 0 \,, & G_2^T + F_3 J^T = 0 \,, \end{array} \tag{78}$$

whose solution gives an $S \in Sp(4n, \mathbb{R})$ of the form

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & (\mathbb{1}_n - J^T) G^{-T} & 0 \\ 0 & J & 0 & G \\ \hline -G^T J^{-T} & 0 & \mathbb{1}_n & 0 \\ 0 & G^{-1} J (J - \mathbb{1}_n) & 0 & G^{-1} J G \end{array} \right] , \tag{79}$$

with G being an arbitrary matrix $G \in Gl(n, \mathbb{R})$. As a consequence of this fact, and because the eigenvalues of J are assumed to be different from 1, s_2, s_3 and s_4 are also non-singular. This is important because it shows that in choosing S as in the canonical form (79) we are not restricting generality: The value of s_2 can always be absorbed into the definition of the covariance matrix γ_E of $\hat{\rho}_E$ by writing

$$\gamma_E = s_2^{-1} \, Y_c \, s_2^{-T} \,, \tag{80}$$

(see also Appendix D). Taking this into account, we can conclude that Eq. (79) provides an explicit demonstration of Lemma 1 for channels of class (i) with non-singular X.

Since Φ_c is fully determined by X_c and Y_c, the above expressions show that the action of Φ_c on the input state does not depend on the choice of G. As a matter of fact, the latter can be seen as a Gaussian unitary operation $\hat{U}_G$ characterized by the n-modes symplectic transformation $Sp(2n, \mathbb{R})$,

$$\Delta_G = \left[\begin{array}{c|c} G^T & 0 \\ \hline 0 & G^{-1} \end{array}\right], \tag{81}$$

applied to final state of the environment after the interaction with the input, i.e., $\tilde{\Phi}_G = \hat{U}_G \tilde{\Phi} \hat{U}_G^T$, where $\tilde{\Phi}$ is the *weak complementary map* for $G = \mathbb{1}_n$, and $\tilde{\Phi}_G$ is the weak complementary map in presence of $G \neq \mathbb{1}_n$ – see the next section for details. Since the relevant properties of a channel (e.g., weak degradability [16, 17]) do not depend on local unitary transformations to the input/output states, without loss of generality, we can consider $G = -J$ and the canonical form for $S \in Sp(4n, \mathbb{R})$ assumes the following simple expression

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & \mathbb{1}_n - J^{-T} & 0 \\ 0 & J & 0 & -J \\ \hline \mathbb{1}_n & 0 & \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J & 0 & J \end{array}\right]. \tag{82}$$

The possibility of constructing different, but unitarily equivalent, canonical forms for S is discussed in Appendix D.

3 Weak degradability

Among other properties, the unitary dilations introduced in Section 2 are useful to define *complementary* or *weak complementary* channels of a given BGC Φ. These are defined as the CPT map $\tilde{\Phi}$ which describes the evolution of the environment under the influence of the physical operation describing the channel [16, 17], i.e.,

$$\tilde{\Phi}(\hat{\rho}) := \mathrm{Tr}_S[\hat{U}(\hat{\rho} \otimes \hat{\rho}_E)\hat{U}^\dagger], \tag{83}$$

where $\hat{\rho}$, $\hat{\rho}_E$ and $\hat{U}$ are defined as in Eq. (20), but the partial trace is now taken over the system modes.

Specifically, if the state $\hat{\rho}_E$ we employed in constructing the unitary dilation of Eq. (20) is pure, then the map $\tilde{\Phi}$ is said to be the *complementary* of Φ and, up to partial isometry, it is unique [27, 28, 29, 30]. Otherwise it is called *weak complementary* [16, 17]. Since in Eq. (20) the state $\hat{\rho}_E$ is Gaussian and $\hat{U}$ is a unitary Gaussian transformation, one can verify that $\tilde{\Phi}$ is also BGC[6]. Expressing the Gaussian unitary transformation $\hat{U}$ in terms of its symplectic matrix S of Eq. (22) the action of $\tilde{\Phi}$ is fully characterized by the following mapping of the covariance matrices γ of $\hat{\rho}$, i.e.,

$$\tilde{\Phi}: \ \gamma \longmapsto \ s_3 \gamma s_3^T + s_4 \gamma_E s_4^T, \tag{84}$$

[6]In general however, it will not map the n input modes into n output modes. Instead it will transform them into ℓ modes, with ℓ being the number of modes assumed in the unitary dilation (20).

which is counterpart of the transformations (16) and (24) that characterize Φ. The channel $\tilde{\Phi}$ is then described by the matrices $\tilde{X} = s_3^T$ and $\tilde{Y} = s_4 \gamma_E s_4^T$ which, according to the symplectic properties (23), satisfy the condition

$$\tilde{Y} \geqslant i\tilde{\Sigma} \qquad \text{with} \quad \tilde{\Sigma} := \sigma_{2\ell}^E - \tilde{X}^T \sigma_{2n} \tilde{X} \,. \tag{85}$$

The relations between Φ and its weak complementary $\tilde{\Phi}$ contain useful information about the channel Φ itself. In particular we say that the channel Φ is *weakly degradable* (WD) while $\tilde{\Phi}$ is *anti-degradable* (AD), if there exists a CPT map $\mathcal{T}$ which, for all inputs $\hat{\rho}$, allows one to recover $\tilde{\Phi}(\hat{\rho})$ by acting on the output state $\Phi(\hat{\rho})$, i.e.

$$\mathcal{T} \circ \Phi = \tilde{\Phi} \,. \tag{86}$$

Similarly, one says that Φ is AD and $\tilde{\Phi}$ is WD if there exists a CPT map $\bar{\mathcal{T}}$ such that

$$\bar{\mathcal{T}} \circ \tilde{\Phi} = \Phi \,. \tag{87}$$

Weak degradability [16, 17] is a property of quantum channels Φ generalizing the *degradability property* introduced in Ref. [27]. The relevance of weak-degradability analysis stems from the fact that it allows one to simplify the quantum capacity scenario. Indeed, it is known that AD channels have zero quantum capacity [16, 17], while WD channels with $\hat{\rho}_E$ pure are degradable and thus admits a single letter expression for this quantity [27]. A complete weak-degradability analysis of single mode bosonic Gaussian channels has been provided in Ref. [16, 17]. Here we generalize some of these results to $n > 1$.

3.1 A criterion for weak degradability

In this section we review a general criterion for degradability of BGCs which was introduced in Ref. [15], adapting it to include also weak degradability. Before entering the details of our derivation, however, it is worth noticing that generic multi-mode Gaussian channels are neither WD nor AD. Consider in fact a WD single-mode Gaussian channel Φ having no zero quantum capacity $Q > 0$ (e.g., a beam-splitter channel with transmissivity $> 1/2$). Define then the two mode channel $\Phi \otimes \tilde{\Phi}$ with $\tilde{\Phi}$ being its weak complementary defined in [16, 17]. This is Gaussian since both Φ and $\tilde{\Phi}$ are Gaussian. The claim is that $\Phi \otimes \tilde{\Phi}$ is neither WD nor AD. Indeed, its weak complementary can be identified with the map $\tilde{\Phi} \otimes \Phi$. Consequently, since $\Phi \otimes \tilde{\Phi}$ and $\tilde{\Phi} \otimes \Phi$ differ by a permutation, they must have the same quantum capacity Q'. Therefore if one of the two is WD than *both* of them must also be AD. In this case Q' should be zero which is clearly not possible given that $Q' \geqslant Q$. In fact, one can use $\Phi \otimes \tilde{\Phi}$ to reliably transfer quantum information by encoding it into the inputs of Φ. In this respect the possibility of classifying (almost) all single-mode Gaussian maps in terms of weak degradability property turns to be rather a remarkable property. We now turn to investigating the weak degradability properties of multi-mode bosonic Gaussian channels deriving a criterion that will be applied in Sec. 4.1 for studying in details the two-mode channels case.

Consider a n-mode bosonic Gaussian channel Φ characterized the unitary dilation (20) and its weak complementary $\tilde{\Phi}$ (83). Let $\{X, Y\}$, $\{\tilde{X}, \tilde{Y}\}$ be the matrices which define such channels. For the sake of simplicity we will assume X and $\tilde{X}$ to be non-singular, $X, \tilde{X} \in Gl(2n, \mathbb{R})$. Examples of such maps are for instance the channels of class (i) with X non-singular described in Sec. 2.5. Adopting in fact the canonical form (82) for S we have that

$$X = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J^T \end{bmatrix} \tag{88}$$

with all the eigenvalues of J being different from 1.

Suppose now that Φ is weakly degradable with $\mathcal{T}$ being the connecting CPT map which satisfies the weak degradability condition (86). As in Refs. [16, 17] we will focus on the case in which $\mathcal{T}$ is BGC and described by matrices $\{X_{\mathcal{T}}, Y_{\mathcal{T}}\}$. Under these hypothesis the identity (86) can be simplified by using the composition rules for BGCs given in Eq. (18). Accordingly, one must have

$$\begin{aligned} X_{\mathcal{T}} &= X^{-1}\tilde{X}\,, \\ Y_{\mathcal{T}} &= \tilde{Y} - X_{\mathcal{T}}^T Y X_{\mathcal{T}}\,. \end{aligned} \tag{89}$$

These definitions must be compatible with the requirement that $\mathcal{T}$ should be a CPT map which transforms the n system modes into the ℓ environmental modes, i.e.,

$$Y_{\mathcal{T}} \geqslant i\left(\sigma_{2\ell}^E - X_{\mathcal{T}}^T \sigma_{2n} X_{\mathcal{T}}\right)\,. \tag{90}$$

Combining the expressions above, one finds the following weak-degradability condition for n-mode bosonic Gaussian channels [15], i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E \geqslant 0\,. \tag{91}$$

In order to obtain the anti-degradability condition (87), it is sufficient to swap $\{X, Y\}$ with $\{\tilde{X}, \tilde{Y}\}$ and the system commutation matrix σ_{2n} with $\sigma_{2\ell}^E$, in Eq. (91), i.e.,

$$Y - X^T \tilde{X}^{-T}(\tilde{Y} + i\sigma_{2\ell}^E)\tilde{X}^{-1}X + i\sigma_{2n} \geqslant 0\,. \tag{92}$$

Equations (91) and (92) are strictly related. Indeed since

$$\begin{aligned} &Y - X^T \tilde{X}^{-T}(\tilde{Y} + i\sigma_{2\ell}^E)\tilde{X}^{-1}X + i\sigma_{2n} \\ &\quad = -X^T\tilde{X}^{-T}\left(\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E\right)\tilde{X}^{-1}X\,, \end{aligned} \tag{93}$$

equation (92) corresponds to reverse the sign of the inequality (91), i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E \leqslant 0\,. \tag{94}$$

Hence to determine if Φ is a weakly degradable or anti-degradable channel, it is then sufficient to study the positivity of the Hermitian matrix

$$W := \tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E\,. \tag{95}$$

In the case in which $\ell = n$ this can be simplified by reminding that an Hermitian $2n \times 2n$ matrix W partitioned as

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^\dagger & W_3 \end{bmatrix} \tag{96}$$

with W_i being $n \times n$ matrices is semi-positive definite if and only if

$$W_1 \geqslant 0 \;\text{ and }\; W_3 - W_2^\dagger W_1^{-1} W_2 \geqslant 0\,, \tag{97}$$

the right hand side being the Schur complement of W (see, e.g., page 472 in Ref. [26]). Using this result and the canonical form (82), Eq. (91) can be written as in Eq. (97) with

$$\begin{aligned} W_1 &= (\mathbb{1}_n - J^{-T})^{-1} Y_1 (\mathbb{1}_n - J^{-1})^{-1} - Y_1 \\ W_2 &= i(J^{-T} - 2\mathbb{1}_n) - Y_2(J^{-T} - \mathbb{1}_n) - (\mathbb{1}_n - J^{-T})^{-1} Y_2 \\ W_3 &= Y_3 - (J^{-1} - \mathbb{1}_n) Y_3 (J^{-T} - \mathbb{1}_n)\,, \end{aligned} \tag{98}$$

and

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}. \tag{99}$$

For the anti-degradability condition (92) simply replace $[\geqslant]$ with $[\leqslant]$ in Eq. (97).

4 Two-mode bosonic Gaussian channels

Here we consider a particular case of n-mode bosonic Gaussian channel analysis above, namely, the case of $n = 2$. This is by no means such a special case as one might at first be tempted to think since any n-mode channel can always be reduced to single-mode and two-mode parts [20]. For two-mode channels the interaction part and the noise term of a generic two-mode bosonic Gaussian channel, X and Y, respectively, are 4×4 real matrices. Particularly, we will focus on two-mode channels Φ which have non-singular X and belong to the class (i) of Theorem 1 (i.e., which have $r = \text{rank}[\Sigma] = 4$), like in Sec. 2.5. These maps can be grasped in terms of a unitary dilation of the form (82) coupling the two system bosonic modes with two additional (environmental) modes, where J is a 2×2 real Jordan block. In order to characterize this large class of two-mode BGCs, one has to examine only three possible forms of J:

- Class A: This corresponds to taking a diagonalizable Jordan block, that is,

  $$J := J_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \tag{100}$$

  where a and b are real nonzero numbers. It represents the trivial case of a two-mode bosonic Gaussian channel, whose interaction term does not couple the two modes. Actually, we call it of class A_1 if $a \neq b$ and of class A_2 otherwise.

- Class B: This is to take J as a non-diagonalizable matrix with a nonzero real eigenvalue a with double algebraic multiplicity (but with geometric multiplicity equal to one), i.e.

$$J := J_1 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} . \tag{101}$$

In this case the Jordan block is called defective [26]. Here, a noisy interaction between the bosonic system and the environment, coupling the two system modes, is switched on.

- Class C: Here the real Jordan block J has complex eigenvalues, i.e.

$$J := J_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} , \tag{102}$$

with $b \neq 0$; the eigenvalues of J are $a \pm ib$. Again, the two system modes are coupled by the noisy interaction with the environment through the presence of the element b.

In order to explicit the form of $Y = s_2 \gamma_E s_2^T$, with s_2 being defined as in Eq. (82), we consider a generic two-mode covariance matrix in the so-called standard form [32] for the environmental initial covariance matrix γ_E, i.e.

$$\gamma_E = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} , \tag{103}$$

where

$$\Gamma_{1,2} := \begin{bmatrix} x & z_{-,+} \\ z_{-,+} & y \end{bmatrix} , \tag{104}$$

and x, y, $z_{+,-}$ are real number satisfying $x + y \geqslant 0$, $xy - z_-^2 \geqslant 1$ and $x^2y^2 - y^2 - x^2 + (z_- z_+ - 1)^2 - xy(z_-^2 + z_+^2) \geqslant 0$ because of the uncertainty principle. More generally, one can apply a generic two-mode (symplectic) squeezing operator $V(\epsilon)$ to the environmental input state, i.e.,

$$\gamma_E' = V(\epsilon) \gamma_E V(\epsilon)^T \tag{105}$$

where

$$V(\epsilon) = \begin{bmatrix} R^{-T} & 0 \\ 0 & R \end{bmatrix} , \quad R = \begin{bmatrix} c + hs & -qs \\ -qs & c - hs \end{bmatrix} , \tag{106}$$

and $c = \cosh(2r)$, $s = \sinh(2r)$, $h = \cos(2\phi)$, $q = \sin(2\phi)$ and $\epsilon = re^{2i\phi}$ being the squeezing parameter [32]. Finally, it is interesting to study how the canonical forms of two-mode BGCs compose under the product. A simple calculation shows that the following rules apply

$\circ$	A	B	C
A	A	A_1/B	$A_1/B/C$
B	A_1/B	A_2/B	$A_1/B/C$
C	$A_1/B/C$	$A_1/B/C$	A/C

.

In this table, for instance, the element on row 1 and column 1 represents the class (i.e., A) associated to the composition of two channels of the same class A. Note that the canonical form of the products with a "coupled" channel (i.e., with B or C) is often not uniquely defined. For instance, composing two class B (with

$$(J_1)_i = \begin{bmatrix} a_i & 1 \\ 0 & a_i \end{bmatrix} \tag{107}$$

with $i = 1, 2$) channels will give us either a class A_2 channel (if $a_1 + a_2 = 0$) or a class B channel (if $a_1 + a_2 \neq 0$). Composition rules analogous to those reported above have been analyzed in details for the one-mode case in Ref. [17]. In the following we will study the weak-degradability properties of these three classes of two-mode Gaussian channels.

4.1 Weak-degradability properties

The weak-degradability conditions in Eqs. (97) become

$$\Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1}) \geqslant 0 \tag{108}$$

and

$$\begin{aligned} & J\Gamma_2 J^T - (\mathbb{1}_2 - J)\Gamma_2(\mathbb{1}_2 - J^T) \\ & \quad -(J^{-1} - 2\mathbb{1}_2)\left[\Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1})\right]^{-1}(J^{-T} - 2\mathbb{1}_2) \geqslant 0\,. \end{aligned} \tag{109}$$

In the same way, the anti-degradability is obtained when both these quantities are non-positive. As concerns the environmental initial state of the unitary dilation, one can consider a generic two-mode state as in Eq. (105). On one hand, we find that, if $[J, R] = 0$, this two-mode squeezing transformation $V(\epsilon)$ can be simply "absorbed" in local symplectic operations to the output states and then it does not affect the weak-degradability properties. On the other hand, if $[J, R] \neq 0$, we find numerically that the introduction of correlations between the two environmental modes contrasts with the presence of (anti-) weak-degradability features. Therefore, one can consider the particular case in which the environment is initially in a state with a symmetric covariance matrix γ_E as in Eq. (103) with $x = y = 2N + 1$ and $z_- = z_+ = 0$ where $N \geqslant 0$. In this case $\gamma_E = (2N + 1)\mathbb{1}_2$ corresponds to a thermal state of two uncoupled environmental modes with the same photon average number N and it is possible to see the results above easily through analytical details. In fact, we study analytically the positivity condition in Eq. (91) in the three possible forms of the real Jordan block J_i.

In the uncoupled case J_0 as in Eq. (100), substituting in Eq. (91), we find that these two-mode bosonic Gaussian channels are WD if $a, b \geqslant 1/2$ and AD for $a, b \leq 1/2$ (any $N \geqslant 0$). In other words, in the case of two uncoupled modes, the weak-degradability properties can be derived from the results for one-mode bosonic Gaussian channels: tensoring two WD (AD) one-mode Gaussian channels with WD (AD) one-mode Gaussian channels yield two-mode Gaussian channels which are WD (AD).

In the case of defective J, i.e., J_1 as in Eq. (101), corresponding to noisy interaction coupling the two system modes, substituting in Eq. (91), we find that, on one hand, these two-mode bosonic Gaussian channels are WD if $a > 1$ and

$$N \geqslant N_1 := \frac{1}{2}\left[-1 + \frac{1}{2}\frac{|2a-1|}{\sqrt{a(a-1)}}\right] . \tag{110}$$

On the other hand, it is AD if $a < 0$ and $N \geqslant N_1$ (see Fig. 2). Note that the defective Jordan blocks are not usually stable with respect to perturbations [20]. Indeed, we find numerically that, applying proper two-mode squeezing transformations to the environmental input, these weak-degradability conditions reduce to the decoupled case ones. In Fig. 1 we consider, for simplicity, a symmetric environmental initial state γ'_E as in Eq. (105) with $x = y$, $z_- = 0$ and $\epsilon = r$, and we plot the relation between x, z_+ and the minimum value of r such that $J := J_1$ reduces to $J := J_0$ corresponding to the decoupled case. One realizes that a squeezing parameter r close to 1 is enough to decouple the two modes representing the system, carrying quantum information. Moreover, let us point out that this squeezing threshold (r) increases slightly with the presence of correlations (z_+) while decreases when increasing the level of noise (x) in the initial environmental state γ'_E.

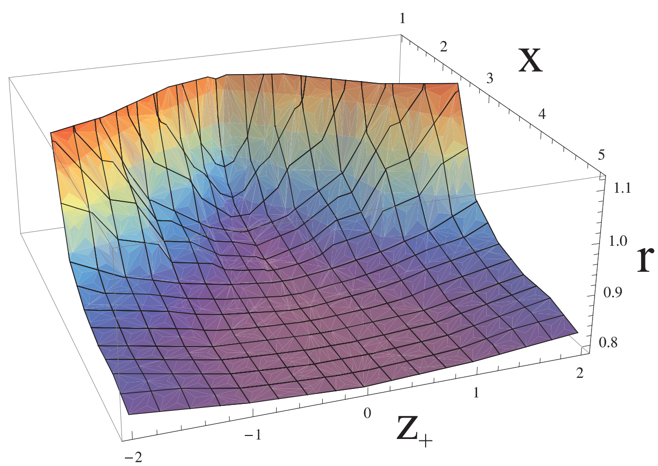

Figure 1: Relation between the parameters x, z_+ and the minimum value of r in the initial environmental state such that the two-mode channel with $X = \mathbb{1}_2 \oplus J_1$ reduces to the decoupled case $X' = \mathbb{1}_2 \oplus J_0$ with the same interaction parameter a for the two system modes.

Finally, in the case of real Jordan block with complex eigenvalues, i.e., J_2 as in Eq. (102), the corresponding two-mode bosonic Gaussian channels are WD if $a > 1/2$ and

$$N \geqslant N_2 := \frac{1}{2}\left[-1 + \left(1 + \frac{4b^2}{(1-2a)^2}\right)^{1/2}\right] . \tag{111}$$

while they are AD if $a < 1/2$ and $N \geqslant N_2$ (see Fig. 2). In both of these cases (real and complex eigenvalues), in which the interaction term couples the two bosonic

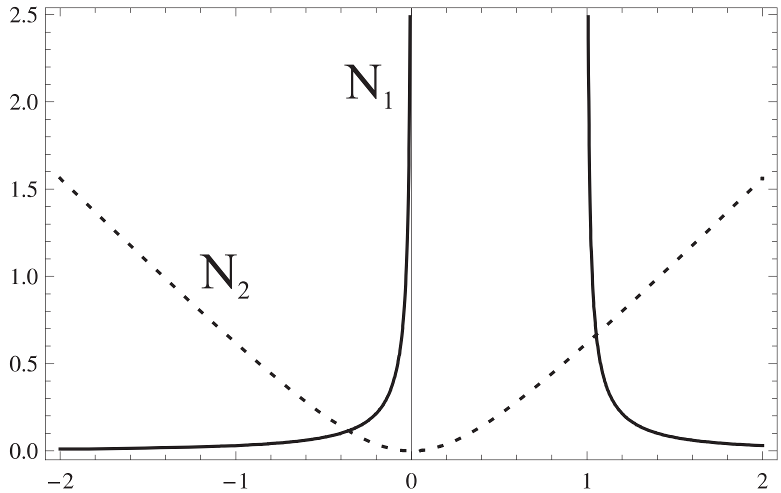

Figure 2: In continuous line we report N_1 as function of a in the case of J_1. For $N \geqslant N_1$ the map is WD if $a > 1$ and AD if $a < 0$. In dashed line we plot N_2 as function of b when $a = 0$ in the case of J_2. For $N \geq N_2$ the channel is (AD) WD if $a > 1/2$ ($a < 1/2$).

modes, there is the (apparently) counter-intuitive fact that above a certain environmental noise threshold the weak-degradability features appear, while for one-mode bosonic Gaussian channels they do not depend on the initial state of the environment. Actually, one would expect at most that, when the level of the environmental noise increases, the coherence progressively decreases until to be destroyed. It would mean that it becomes more and more difficult to recover the environment (system) output from the system (environment) output after the noisy evolution. However, the things go the other way around when multi-mode bosonic Gaussian channels are considered.

4.2 Channels with zero quantum capacity

Analogously to Ref. [17] where the one-mode case is investigated, one can enlarge (other than the AD maps) the class of two-mode BGCs with $Q = 0$, composing a generic channel with an AD one. First of all, consider a channel Φ as in Section 2.3, but being AD (not necessarily minimal noise), then the maps Φ', defined in Eq. (63), have zero quantum capacity, i.e., they cannot be used to transfer quantum information. For instance, one can choose $\gamma_E = (2N_c+1)\mathbb{1}_n$, i.e., the environmental initial state of the map Φ is a multi-mode thermal state with N_c being the average photon number for each mode, such that Φ is AD or simply with zero capacity; therefore, for any $\gamma'_E \geqslant \gamma_E = (2N_c + 1)\mathbb{1}_n$, as in Eq. (65), the map Φ' of Eq. (63) has $Q = 0$. Particularly for $n = 2$, using these observations and choosing N_c equal to either N_1 (and $a < 0$) or N_2 (and $a < 1/2$) as in Eqs. (110) and (111), one obtains that for $X = \mathbb{1}_2 \oplus J_{1,2}$ and $Y' = s_2\gamma'_E s_2^T$ [with s_2 as in Eq. (82)] the resulting channel Φ' has always zero capacity. In this way, one extends considerably the set of two-modes maps with zero capacity, other than the very particular cases of two-mode environmental thermal states studied above and shown in Fig. 2. For instance, two-mode squeezing can be applied to the thermal state γ_E including not only states with $N > N_c$ but also with not trivial two-mode correlations such that $\gamma'_E \geqslant (2N_c+1)\mathbb{1}_2$. Therefore, just considering this last simple inequality one includes so a larger set of maps that have zero quantum capacity.

Moreover, we observe that, according to composition rules above, the combination $\Phi = \Phi_{II} \circ \Phi_I$ of two channels Φ_I and Φ_{II} of class A_2 and C, respectively, with Jordan blocks J_I as in Eq. (100) with $a_I = b_I$ and J_{II} as in Eq. (102) with a_{II} and $b_{II} \neq 0$, gives $J = a_I J_{II}$ which is in the class C. Now, since we have $N_1 \geqslant 0$, $N_2 \geqslant 0$ and assuming $a_I \leqslant 1/2$, the channel Φ_I is AD and the resulting channel Φ must have $Q = 0$. Varying the parameters but keeping the product $a_I a_{II} = a$ and $a_I b_{II} = b$ fixed, the parameter N can assume any value satisfying the inequality

$$N \;\; \geqslant \;\; \frac{1}{4}\left[\left(\frac{5(1-4a+8a^2+8b^2)}{b^2+(a-1)^2}\right)^{1/2} - 2\right] . \tag{112}$$

Note that a_I has been chosen equal to $1/2$ and Φ_I corresponds to two uncoupled beam-splitter maps with transmissivity $1/2$. We can therefore conclude that all channels of the form C with N as in Eq. (112) have zero quantum capacity – see Fig. 3.

Consider now the composition $\Phi = \Phi_{II} \circ \Phi_I$ of two channels Φ_I and Φ_{II} of class C and A_2 (i.e., in the opposite order with respect to above), respectively, with Jordan blocks J_I as in Eq. (102) with a_I and $b_I \neq 0$ and J_{II} as in Eq. (100) with $a_{II} = b_{II}$, giving $J = a_{II} J_I$ which is in the class C. As before, since we have $N_1 \geqslant 0$, $N_2 \geqslant 0$ and assuming again $a_{II} \leqslant 1/2$, the channel Φ_2 is AD and the resulting channel has $Q = 0$. Varying the parameters but keeping the product $a_I a_{II} = a$ and $b_I a_{II} = b$ fixed, the parameter N can assume any value satisfying the inequality

$$N \;\; \geqslant \;\; \frac{1}{4}\left[\left(\frac{(1+4a^2+4b^2)(1-4a+8a^2+8b^2)}{4(b^2+(a-1)^2)(a^2+b^2)}\right)^{1/2} - 2\right] , \tag{113}$$

where again a_{II} is chosen equal to $1/2$. Again we can conclude that all class C channels with N as in Eq. (113) have zero quantum capacity. However, notice that the constraint in Eq. (113) is an improvement with respect to the constraint of Eq. (112) – see Fig. 3.

5 Conclusions

In this work, we have presented a complete analysis of generic multi-mode Gaussian channels by proving a unitary dilation theorem and by finding their canonical form. This is a simple form that can be achieved for any Gaussian quantum channel, as a convenient starting point for various considerations. For instance, it allows us to simplify the analysis of the weak-degradability properties of multi-mode bosonic Gaussian channels. Minimal output entropies, or quantum and classical information capacities and other difficult questions might be tackled using the canonical form of multi-mode Gaussian channels shown in this paper. Here, we investigated in details

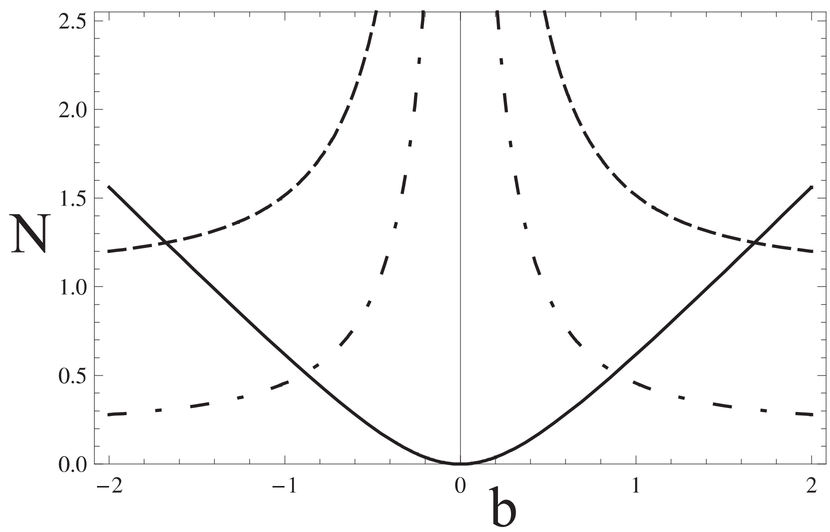

Figure 3: The continuous line depicts plot N_2 as in Eq. (111) versus b, with $a = 1$ in J_2 of Eq. (102). For $N \geq N_2$ the channel is WD (AD) if $a > 1/2$ ($a < 1/2$). The dashed line refers to the bound in Eq. (112), while the dashed-dot line to the one in Eq. (113); above these bounds the class C map is WD but with $Q = 0$. Note that Eq. (113) is an improvement with respect to the constraint of Eq. (112). Similar bounds can be obtained in the case $a < 1/2$, enlarging the group of AD maps with other channels with $Q = 0$.

the two-mode scenario that is relevant since any n-mode channel can always be reduced to single-mode and two-mode parts [20]. Furthermore, the results of this paper could play a basic role in characterizing the efficiency of continuous-variables quantum information processing, quantum communication and quantum key distribution protocols.

6 Acknowledgements

F.C. and V.G. thank the Quantum Information research program of Centro di Ricerca Matematica Ennio De Giorgi of Scuola Normale Superiore for financial support. J.E. acknowledges the EPSRC, the QIP-IRC, the EU (QAP, COMPAS), Microsoft Research, and the EURYI Award Scheme for financial support. A.S.H. acknowledges support of RFBR grant 06-01-00164-a and the RAS program “Modern problems of theoretical mathematics”.

A Proof of Lemma 1

Note that it does not restrict generality to take $\sigma_{2\ell}^E = \sigma_{2\ell}$, as this can always be accompanied by an appropriate similarity transform. Our problem at hand of extending a symplectic form is then equivalent to the following problem: Suppose we are given column vectors $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ from $\mathbb{R}^{2(n+\ell)}$ that satisfy

$$e_j^T \sigma_{2(n+\ell)} e_k = 0, \tag{114}$$
$$f_j^T \sigma_{2(n+\ell)} f_k = 0, \tag{115}$$
$$e_j^T \sigma_{2(n+\ell)} f_k = \delta_{j,k}, \tag{116}$$

for $j, k = 1, \cdots, n$. The procedure continues by identifying vectors e_{n+1} and f_{n+1} such that $e_{n+1}^T \sigma_{2(n+\ell)} f_{n+1} = 1$ and

$$e_{n+1}^T \sigma_{2(n+\ell)} w = f_{n+1}^T \sigma_{2(n+\ell)} w = 0 \tag{117}$$

for all

$$w \in W_n := \text{span}(e_1, \cdots, e_n, f_1, \cdots, f_n). \tag{118}$$

Now define

$$W_n^\perp = \{w : w^T \sigma_{2(n+\ell)} v = 0 \, \forall v \in W_n\}. \tag{119}$$

It is now not difficult to see that $W_n \cap W_n^\perp = \{0\}$ and $\mathbb{R}^{2(n+\ell)} = W_n \oplus W_n^\perp$: Suppose that the vector v has $v^T \sigma_{2(n+\ell)} e_j =: \alpha_j$ and $v^T \sigma_{2(n+\ell)} f_j =: \beta_j$ for $j = 1, \cdots, n$. Then

$$v = \left[\sum_{j=1}^n (-\alpha_j f_j + \beta_j e_j)\right] + \left[v + \sum_{j=1}^n (\alpha_j f_j - \beta_j e_j)\right], \tag{120}$$

where the first term is element of W_n and the second of $W_n^\perp$. Following a symplectic Gram-Schmidt procedure, the symplectic basis can hence be completed, which is equivalent to extending the matrices s_1 and s_2 to a symplectic

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \in Sp(2(n+\ell), \mathbb{R}). \tag{121}$$

B Derivation of Eq. (35)

Here we show that Eq. (35) admits solution for s_2' as in Eq. (41). In fact, assuming $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n}$ with σ_{2n} as in Eq. (1), one has

$$\begin{aligned} s_2' \, \sigma_{4n}^E \, (s_2')^T - \Sigma' &= \left[\begin{array}{c|c} K^{-1} & O^T A \end{array}\right] \left[\begin{array}{c|c} \sigma_{2n} & 0 \\ \hline 0 & \sigma_{2n} \end{array}\right] \left[\begin{array}{c} K^{-T} \\ \hline A^T O \end{array}\right] - \Sigma' \\ &= K^{-1} \sigma_{2n} K^{-T} + O^T A \, \sigma_{2n} \, A^T O - \Sigma' \\ &= K^{-1} \left(K \Sigma' K^T + B\right) K^{-T} + O^T A \, \sigma_{2n} \, A^T O - \Sigma' \\ &= K^{-1} B K^{-T} + O^T A \, \sigma_{2n} \, A^T O \\ &= O \left(M^{1/2} B M^{1/2} + A \, \sigma_{2n} \, A^T\right) O^T, \end{aligned} \tag{122}$$

where we used Eq. (40) to write $\sigma_{2n} = K \Sigma' K^T + B$, with B being the $2n \times 2n$ matrix

$$B := \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline 0 & -\mathbb{1}_{n-r/2} \end{array} & 0 \end{array}\right]. \tag{123}$$

The identity (35) finally follows by noticing that the last term in Eq. (122) cancels since $M^{1/2} B = B M^{1/2} = B$ and $A \, \sigma_{2n} \, A^T = -B$.

C Properties of the environmental states

In this appendix we first give an explicit derivation of Eq. (43). Then we analyze in details the property of the state $\hat{\rho}_E$ associated with the covariance matrix γ_E defined be the Eqs. (45) and (46). Replacing Eq. (41) into Eq. (26), we get

$$
\begin{aligned}
\mathbb{1}_{2n} = s_2'\,\gamma_E\,(s_2')^T &= \left[\begin{array}{c|c} K^{-1} & O^T A \end{array}\right] \left[\begin{array}{c|c} \alpha & \delta \\ \hline \delta^T & \beta \end{array}\right] \left[\begin{array}{c} K^{-T} \\ \hline A^T O \end{array}\right] \\
&= K^{-1}\,\alpha\,K^{-T} + O^T\,A\,\delta^T\,K^{-T} + K^{-1}\,\delta\,A^T\,O + O^T\,A\,\beta\,A^T\,O \\
&= O^T\left(M^{1/2}\,\alpha\,M^{1/2} + A\,\delta^T\,M^{1/2} + M^{1/2}\,\delta\,A^T + A\,\beta\,A^T\right)O\,,
\end{aligned}
$$

which leads to

$$
M^{-1} = \alpha + M^{-1/2}A\,\delta^T + \delta\,A^T M^{-1/2} + M^{-1/2}\,A\,\beta\,A^T M^{-1/2}\,, \tag{124}
$$

and hence to Eq. (43) by the fact $M^{-1/2}A = A^T M^{-1/2} = A = A^T$. Such an equation admits the solution given in Eqs. (45) and (46). Explicitly this corresponds to the $4n \times 4n$ covariance matrix γ_E of the form

$$
\left[\begin{array}{c|c|c|c}
\begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & 0 & 0 & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} \\ \hline
0 & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & 0 \\ \hline
0 & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & 0 \\ \hline
\begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & 0 & 0 & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array}
\end{array}\right]
$$

where for easy of notation $\mathbb{1} := \mathbb{1}_{n-r/2}$. By looking at the structure of this covariance matrix, one realizes that it is composed by two independent sets formed by r and $2n-r$ modes, respectively. The first set describes $r/2$ thermal states characterized by the matrices μ^{-1} which have been purified adding further $r/2$ modes. The second set instead describes a collection of $2(n-r/2) = 2n-r$ modes prepared in a pure state formed by $n-r/2$ independent pairs of modes which are entangled. By reorganizing its rows and columns this can be cast into the simpler form

$$
\gamma_E \;=\; \begin{array}{l}
\left[\begin{array}{c|c}
\begin{array}{c|c} \bar{\mu}^{-1} & f(\bar{\mu}^{-1}) \\ \hline f(\bar{\mu}^{-1}) & \bar{\mu}^{-1} \end{array} & 0 \\ \hline
0 & \begin{array}{c|c} \xi\mathbb{1}_{2n-r} & f(\xi\mathbb{1}_{2n-r}) \\ \hline f(\xi\mathbb{1}_{2n-r}) & \xi\mathbb{1}_{2n-r} \end{array}
\end{array}\right]
\end{array}
\begin{array}{l} \}\,r \\ \}\,r \\ \}\,2n-r \\ \}\,2n-r\,, \end{array} \tag{125}
$$

where we used $\bar{\mu}$ to indicate the $r \times r$ matrix $\bar{\mu} = \mu \oplus \mu$.

C.1 Solution for $\ell_{\text{pure}} = 2n - r'/2$ environmental modes

Defining r' as in Eq. (57) we choose the environmental commutation matrix to be $\sigma_{2\ell}^E = \sigma_{2n} \oplus \sigma_{2n-r'}$ with σ_{2n} and $\sigma_{2n-r'}$ as in Eq. (1). A unitary dilation with $\ell_{\text{pure}} =$

$2n - r'/2$ environmental modes in a pure state is obtained by having $s_2 = Y^{1/2} s_2'$ with s_2' as in Eq. (41). In this case, however, A is a rectangular matrix $2n \times 2(n - r'/2)$ of the form

$$A \;=\; \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} & 0 \end{array}\right] \begin{array}{l} \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2 \\ \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2. \end{array} \tag{126}$$

Similarly, the covariance matrix γ_E can be still expressed as in Eq. (44). In this case, yet, α is a $2n \times 2n$ matrix of block form

$$\alpha = \left[\begin{array}{c|c} \begin{array}{c|c|c} \mathbb{1}_{r'/2} & 0 & 0 \\ \hline 0 & \mu_o^{-1} & 0 \\ \hline 0 & 0 & \xi\mathbb{1}_{n-r/2} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c|c} \mathbb{1}_{r'/2} & 0 & 0 \\ \hline 0 & \mu_o^{-1} & 0 \\ \hline 0 & 0 & \xi\mathbb{1}_{n-r/2} \end{array} \end{array}\right] \begin{array}{l} \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2 \\ \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2, \end{array} \tag{127}$$

where $\xi = 5/4$ and μ_o is the $(r - r')/2 \times (r - r')/2$ diagonal matrix formed by the elements of μ which are strictly smaller than 1. β is the $(2n - r') \times (2n - r')$ matrix

$$\beta = \left[\begin{array}{c|c} \begin{array}{c|c} \mu_o^{-1} & 0 \\ \hline 0 & \xi\mathbb{1}_{n-r/2} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \mu_o^{-1} & 0 \\ \hline 0 & \xi\mathbb{1}_{n-r/2} \end{array} \end{array}\right] \begin{array}{l} \}\, (r-r')/2 \\ \}\, n-r/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2, \end{array} \tag{128}$$

and

$$\delta = \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline f(\mu_o^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}_{n-r/2}) \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline f(\mu_o^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}_{n-r/2}) \end{array} & 0 \end{array}\right] \begin{array}{l} \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2 \\ \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, n-r/2, \end{array} \tag{129}$$

with f as in Eq. (46).

By looking at the structure of this covariance matrix, one realizes that it is composed by three independent pieces. The first one describes a collection of $r'/2$ vacuum states. The second one, in turn, describes $(r - r')/2$ thermal states characterized by the matrices μ_o^{-1} which have been purified by adding further $(r - r')/2$ modes. The third one, finally, reflects a collection of $2(n - r/2) = 2n - r$ modes prepared in a pure state formed by $n - r/2$ independent pairs of modes which are entangled.

C.2 Solution for $\ell = 2n - r/2$ not necessarily pure environmental modes

In this subsection, we present the alternative derivation of a dilation that does not necessarily involve an environment prepared in a pure state. Choosing the commutation matrix $\sigma_{2\ell}^E = \sigma_{2n} \oplus \sigma_{2n-r}$ with σ_{2n} and σ_{2n-r} as in Eq. (1), the matrix s_2' can be still expressed as in Eq. (41). In this case, however, A is a rectangular matrix $2n \times (2n - r)$ of the form

$$A = \left[\begin{array}{c|c} 0 & 0 \\ \cline{2-2} & \mathbb{1}_{n-r/2} \\ \hline 0 & \\ \cline{1-1} \mathbb{1}_{n-r/2} & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \quad . \tag{130}$$

Similarly, γ_E has the block form (44), where α is still the $2n \times 2n$ matrix of Eq. (45), while β and δ are, respectively, the following $(2n-r) \times (2n-r)$ and $2n \times (2n-r)$ real matrices:

$$\beta = \left[\begin{array}{c|c} \xi \mathbb{1}_{n-r/2} & 0 \\ \hline 0 & \xi \mathbb{1}_{n-r/2} \end{array}\right] \begin{array}{l} \}\, n-r/2 \\ \}\, n-r/2, \end{array} \tag{131}$$

$$\delta = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & f(\xi \mathbb{1}_{n-r/2}) \\ \hline 0 & 0 \\ \hline f(\xi \mathbb{1}_{n-r/2}) & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{132}$$

with ξ and f as in Eq. (46). That is,

$$\gamma_E = \left[\begin{array}{cc|cc|cc} \mu^{-1} & 0 & 0 & & 0 & 0 \\ 0 & \xi\mathbb{1} & & & 0 & f(\xi\mathbb{1}) \\ \hline 0 & & \mu^{-1} & 0 & 0 & 0 \\ & & 0 & \xi\mathbb{1} & f(\xi\mathbb{1}) & 0 \\ \hline\hline 0 & 0 & 0 & f(\xi\mathbb{1}) & \xi\mathbb{1} & 0 \\ 0 & f(\xi\mathbb{1}) & 0 & 0 & 0 & \xi\mathbb{1} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2 \\ \}\, n-r/2 \\ \}\, n-r/2\,, \end{array} \tag{133}$$

with $\mathbb{1} = \mathbb{1}_{n-r/2}$. This covariance matrix now consists of two independent parts: The first one describes a collection of $r/2$ thermal states described by the matrices μ^{-1}. The second instead reflects a collection of $2(n - r/2) = 2n - r$ modes prepared in a pure state formed by $n - r/2$ independent couples of modes which are entangled. The covariance matrix given in Theorem 1 can be recovered from the one given above by adding r modes to purify the thermal states μ^{-1}.

D Equivalent unitary dilations

Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \tag{134}$$

and γ_E define a unitary dilation for a bosonic Gaussian channel Φ characterized by matrices X and Y. Then a full class of unitary dilations

$$S' = \begin{bmatrix} s'_1 & s_2 \\ s'_3 & s'_4 \end{bmatrix} \tag{135}$$

can be obtained by taking $\gamma'_E = V\gamma_E V^T$ and

$$s'_1 = s_1\,, \quad s'_2 = s_2 V \quad s'_3 = W s_3\,, \quad s'_4 = W s_4 V\,, \tag{136}$$

with $V \in Sp(2\ell, \mathbb{R})$ and $W \in Sp(2n, \mathbb{R})$ being symplectic transformations of ℓ and n modes respectively. With this choice in fact γ'_E is still a covariance matrix while the conditions (23) and (24) are automatically satisfied. From a physical point of view the symplectic transformations V and W correspond to unitary local operations applied to the environmental input and output states, respectively, by virtue of the metaplectic representation. Consequently, the weak complementary channels $\tilde{\Phi}$ and $\tilde{\Phi}'$ associated with these two representations are unitarily equivalent and the weak-degradability properties one can determine for Φ will be the same when studied for Φ'.

Conversely, let us suppose to have two unitary dilations of Φ, realized with $\ell = n$ environmental modes and characterized by the symplectic matrices S and S' as in Eq. (134) and (135), respectively, with s_i and s'_i being $2n \times 2n$ square matrices. Then it is possible to show that they must be related as in Eq. (136) under the hypothesis that s_2 and s_3 are non-singular. First of all, since Eq. (24) must be satisfied for all the input covariance matrices γ, we have $s_1 = X^T = s'_1$. Define then $V = s_2^{-1} s'_2$ and $W = s'_3 s_3^{-1}$. By using the first of Eq. (23) and exploiting the non-singularity of s_2 one has

$$s_2\, V\, \sigma_{2\ell}^E\, V^T\, s_2^T = s_2\, \sigma_{2n}\, s_2^T \quad \Longrightarrow \quad V\, \sigma_{2n}\, V^T = \sigma_{2n}\,, \tag{137}$$

which implies that V is a symplectic matrix (we are assuming $\sigma_{2\ell}^E = \sigma_{2n}$). Moreover, from the second condition in Eqs. (23) for S and S', we obtain

$$s_2 \sigma s_4^T W^T = s_2 V \sigma s_4'^T\,, \qquad \Longrightarrow \qquad s'_4 = W s_4 V\,, \tag{138}$$

because s_2 is non-singular and V is symplectic. By considering the third condition (23) one then has

$$W(s_3 \sigma_{2n} s_3^T + s_4 \sigma_{2n} s_4^T) W^T = W \sigma_{2n} W^T = \sigma_{2n} \tag{139}$$

which prove that W is a symplectic. Finally, let us observe that the proof above does not use the non-singularity of s_3. Indeed, one can relax this hypothesis and assume more simply that there exists a W such that $s'_3 = W s_3$; from Eqs. (23) W has to still be a symplectic matrix but s_3 and s'_3 may be singular.

As an application of these equivalent unitary dilation results, we can find an alternative canonical form to the one in Sec. 2.5 with the same s_1 and s_4 but with s_2 and s_3 of the following anti-diagonal block form

$$s_j = \begin{bmatrix} 0 & F_j \\ G_j & 0 \end{bmatrix} \tag{140}$$

where, for $j = 2, 3$, F_j, G_j are $n \times n$ real matrices. Imposing Eqs. (23), one obtains the following relations

$$\begin{aligned} J^T - F_2 G_2^T &= \mathbb{1}_n \quad , \quad J'^T - F_3 G_3^T = \mathbb{1}_n , \\ F_2 - F_3^T &= 0 \quad , \quad J G_3^T - G_2 J'^T = 0 \, , \end{aligned} \tag{141}$$

the solution of which provides the following unitary dilation,

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & 0 & -(\mathbb{1}_n - J^T) G_2^{-T} \\ 0 & J & G_2 & 0 \\ \hline 0 & -G_2^{-1}(\mathbb{1}_n - J) & \mathbb{1}_n & 0 \\ G_2^T & 0 & 0 & G_2^T J^T G_2^{-T} \end{array} \right] , \tag{142}$$

where again G_2 is an arbitrary (non-singular) matrix and the eigenvalues of J are assumed to be different from 1. This solution is unitarily equivalent to the one in Eq. (79) by applying $V = -\sigma_{2n}$ and

$$W = \left[\begin{array}{cc} 0 & G_2^{-1} J^{-1} G_2 \\ -G_2^T J^T G_2^{-T} & 0 \end{array} \right] \tag{143}$$

as above.

E The ideal-like quantum channel

Here we consider a quantum channel with $X = \mathbb{1}_{2n}$ but $Y \geqslant 0$ with rank less than $2n$, which can be described in terms of only n additional (environmental) modes. We call it ideal-like quantum channel. Accordingly, the canonical unitary transformation $\hat{U}$ of Eq. (20) will be uniquely determined by a $4n \times 4n$ real matrix $S \in Sp(4n, \mathbb{R})$ of block form in Eq. (22), where s_i are $2n \times 2n$ real matrices. Particularly, $s_1 = s_4 = \mathbb{1}_{2n}$,

$$s_3 = \left[\begin{array}{cc} F_3 & 0 \\ 0 & G_3 \end{array} \right] , \qquad s_2 = \left[\begin{array}{cc} -G_3^T & 0 \\ 0 & -F_3^T \end{array} \right] , \tag{144}$$

with F_3 and G_3 being $n \times n$ real matrices such that $F_3 G_3^T = G_3^T F_3 = 0$, in order to satisfy the symplectic conditions in Eqs. (23). Taking advantage of the freedom in the choice of the unitary dilation shown in Appendix D, the matrix S can be put in the form of Eq. (22) in which $s_1' = s_4' = \mathbb{1}_{2n}$,

$$s_2' = \left[\begin{array}{cc} 0 & 0 \\ 0 & \mathbb{1}_n \end{array} \right] , \qquad s_3' = \left[\begin{array}{cc} -\mathbb{1}_n & 0 \\ 0 & 0 \end{array} \right] , \tag{145}$$

where F_3 is assumed non-singular. In this respect, one uses $V, W \in Sp(2n, \mathbb{R})$ (of App. D) of the following form

$$V = \left[\begin{array}{cc} -F_3 & 0 \\ 0 & -F_3^{-T} \end{array} \right] , \tag{146}$$

and $W = V^{-1}$. Similarly, one can proceed, if G_3 is non-singular, and obtains a similar structure for S as above. As concerns the weak-degradability properties, if one assumes the initial environmental input state as $\gamma_E = \mathrm{diag}(2N+1, 2M+1, 2N+1, 2M+1)$, the eigenvalues of $\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma)X^{-1}\tilde{X} + i\sigma$ are $\{2M, 2(M+1), 2N, 2(N+1)\}$, which are always positive for any $N \geqslant 0$ and $M \geqslant 0$; hence, this channel with γ_E as above is always weakly degradable.

Finally, one may consider another ideal-like channel with $X = \mathbb{1}_{2n}$ and $Y = [(1-\sigma_3)/4]^{\otimes n}$, i.e. $\Phi_{X,Y} = \bigotimes_{i=1}^{n}(B_1)_i$, where the single-mode B_1 channel is defined in Ref. [17] as $X = \mathbb{1}_2$ and $Y = (1-\sigma_3)/4$. Trivially, this multi-mode channel is always WD (like B_1) and is able to transfer a quantum state without decoherence with the maximum quantum capacity (like for the single-mode case [17]).

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