Gaussian-optimized preparation of non-Gaussian pure states

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(Received 26 July 2008; revised manuscript received 23 October 2008; published 13 January 2009)

Non-Gaussian states are highly sought-after resources in continuous-variable quantum optical information processing protocols. We outline a method for the optimized preparation of any pure non-Gaussian state to a given desired accuracy. Our proposal arises from two connected concepts. First, we define the operational cost of a desired state as the largest Fock state required for its approximate preparation. Second, we suggest that this non-Gaussian operational cost can be reduced by judicial application of optimized Gaussian operations. In particular, we identify a minimal core non-Gaussian state for any target pure state, which is related to the core state by Gaussian operations alone. We demonstrate this method for Schrödinger cat states.

DOI: 10.1103/PhysRevA.79.012313 PACS number(s): 03.67.Ac, 42.50.Dv

I. INTRODUCTION

Nonclassical features of quantum states are now interpreted as potential resources ready for exploitation in various quantum information processing tasks. This modern viewpoint is particularly acute in the realm of optical continuous-variable quantum information, where there is a clear distinction between different types of resources. On the one hand, we have the Gaussian states and operations which are readily available and relatively easy to prepare. However, on the other, Gaussian resources alone are not enough and they must be augmented by non-Gaussian resources, with universal quantum computation [1,2] being a crucial example of this. Moreover, during the last three decades, a diverse range of non-Gaussian states of light have been successfully generated: namely, sub-Poissonian states [3], Fock states [4,5], superposition of Fock states [6], single-photon-subtracted states [7–11], single-photon-added states [12], and squeezed two-photon states [13]. In addition, two-mode non-Gaussian states are now understood as potentially useful resources for the enhancement of entanglement and continuous-variable teleportation [14–17], improvement of nonlocal correlations [18,19], and demonstration of the violation of Bell's inequalities with homodyne detectors [20–23], with the first experimental demonstration of this two-mode Gaussian state reported in [24]. So interest in non-Gaussian states and operations is of a fundamental and operational character. However, these necessary states and operations are notoriously difficult to prepare or execute. While this point is widely acknowledged, the question whether Gaussian operations can aid in the construction of non-Gaussian states has been neglected.

In this article, we are interested in the potential application of Gaussian operations to reduce the complexity of non-Gaussian pure-state preparation. We consider this question in the context of experimentally realistic state preparation schemes utilizing photon-subtraction [7,8,10,25,26] and addition [27] techniques, while feasible schemes for multiple-photon-subtracted states were suggested in [28–30]. To be concrete, in the absence of a suitable Hamiltonian, arbitrary non-Gaussian quantum optical states can be approximately constructed using finite high-order nonclassical resources via experimentally feasible photon-subtraction [26] or photon-addition [11,27] methods. These schemes allow the construction of arbitrary finite superpositions of Fock states $\Sigma_{n=0}^{N} c_n|n\rangle$. For example, in [26] such a superposition can be conditionally prepared from a squeezed state subjected to a sequence of $N+1$ displacements interspaced with N-photon subtractions before a final antisqueezing. Similarly, in [27], such a state can be probabilistically prepared from a supply of N single-photon states.

The non-Gaussian resources required in the idealized noiseless execution of these schemes are N—i.e., the number of photon subtractions or the number of single-photon states required to prepare $\Sigma_{n=0}^{N} c_n|n\rangle$. Consequently, we can identify the minimum non-Gaussian resource cost as the minimum number of successive photon subtractions or additions to produce this state. This logic can also be applied to continuous-variable states like $|\psi\rangle = \Sigma_{n=0}^{\infty} c_n|n\rangle$ with a caveat: that the desired state can only be produced approximately by $|\psi^N\rangle = \mathcal{N}\Sigma_{n=0}^{N} c_n|n\rangle$, where $\mathcal{N}$ is a normalization factor. Thus, in this case, the minimum number of photon subtractions or additions required to prepare our desired state to a *sufficient* accuracy, determined by the fidelity $\mathcal{F}(\psi,\psi^N) = |\langle\psi|\psi^N\rangle|^2$, is the operational cost for that state. It should be stressed that this operational cost is not a measure, but a useful ruler for gauging the difficulty of preparing the state. Our method is, therefore, in contrast to approaches which attempt to quantitatively measure the degree of nonclassicality [31–37] or non-Gaussianity possessed by a particular state [38,39].

Our inspiration for the work presented here resides in two simple questions. In the first instance, can unitary Gaussian operations applied to the prepared finite-dimensional state increase the fidelity with the desired continuous-variable target state? In the second, can the applied unitary Gaussian operations help to reduce the number of photon subtractions

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or additions to reach desired fidelity? These questions are motivated by a desire to prepare non-Gaussian states in a manner which minimizes the non-Gaussian resource overhead. If we restrict ourselves to directly preparing truncated versions of $|\psi\rangle$, then we have no freedom in reducing the non-Gaussian resources. This follows from the fact that the fidelity between $|\psi\rangle$ and a truncated approximation scales with the largest Fock state in the latter. Thus, a greater accuracy requires an ever greater number of subtractions or an ever larger Fock-state preparation. Here we suggest an alternative approach to the approximate preparation of $|\psi\rangle$. Instead of directly constructing a truncated version of the target, we advocate the identification and preparation of a minimum core state. This core state will minimize the non-Gaussian resources required to prepare a sufficiently accurate approximation to the target. Moreover, each core is related to the desired target through Gaussian operations alone. Consequently, we are motivated to understand whether Gaussian operations can reduce the accumulative cost of employing ever elaborate non-Gaussian operations. To this end we formulate a criterion to answer this and then use it to optimize the parameters of the associated non-Gaussian operations. In Sec. II we outline our argument and discuss the notion of core states and their preparation. Following this general argument, in Sec. III we illustrate its content and provide evidence of its utility by applying it to the Schrödinger cat states.

II. GAUSSIAN-OPTIMIZED PREPARATION OF NON-GAUSSIAN STATES

A. Identification of core states

Our central problem is this: we want to prepare a very good approximation to the continuous-variable non-Gaussian pure state $|\psi(\lambda)\rangle = \sum_{n=0}^{\infty}\psi_n(\lambda)|n\rangle$, where the parameters $\lambda = (\lambda_1, \ldots, \lambda_M)$ specify a particular state from a family of like states. For example, α labels each possible even parity Schrödinger cat state $|\psi(\alpha)\rangle = \mathcal{N}(|\alpha\rangle + |-\alpha\rangle)$, where $\mathcal{N}$ is a normalization factor. However, we restrict ourselves to Gaussian operations supplemented by photon subtractions or a supply of Fock states. The implementation of the former is considerably easier than the latter; minimization of the non-Gaussian resource is our key priority. Thus, for a given $|\psi(\lambda)\rangle$ we wish to identify the optimal Gaussian parameters that correspond to the smallest number of photon subtractions. To this end, we introduce a family of single-mode core states $|\lambda, r, \alpha, \theta\rangle$ related to the target via

$$|\psi(\lambda)\rangle = \hat{U}(\theta)\hat{D}(\alpha)\hat{S}(r)|\lambda, r, \alpha, \theta\rangle, \tag{1}$$

where $\hat{D}(\alpha)$ and $\hat{S}(r)$ are the single-mode unitary displacement and squeezing operators [40], $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, $\hat{S}(r) = \exp[\frac{r}{2}\{(\hat{a}^\dagger)^2 - \hat{a}^2\}]$, and $\hat{U}(\theta) = e^{i\theta\hat{n}}$ is the phase operator. Conversely, each core state is given by the inverse on the target:

$$|\lambda, r, \alpha, \theta\rangle = \hat{S}(-r)\hat{D}(-\alpha^*)\hat{U}(-\theta)|\psi(\lambda)\rangle. \tag{2}$$

This definition of a corresponding core state allows us to distinguish between two classes of continuous-variable non-Gaussian pure states. The first class is composed of those with a finite-dimensional core,

$$|\psi(\lambda)\rangle = \hat{U}(\theta)\hat{D}(\alpha)\hat{S}(r)\left(\sum_{n=0}^{N} c_n(\lambda)|n\rangle\right), \tag{3}$$

which can be prepared with perfect fidelity by first preparing the finite core before applying the unitary Gaussian operations. Examples of such states include the photon-added coherent state

$$|\psi\rangle = \frac{\hat{a}^\dagger|\alpha\rangle}{\sqrt{1+|\alpha|^2}} = \hat{D}(\alpha)\left(\frac{\alpha^*|0\rangle + |1\rangle}{\sqrt{1+|\alpha|^2}}\right), \tag{4}$$

which can be prepared with unit fidelity by first preparing $c_0|0\rangle + c_1|1\rangle$ and displacing the result. The second, more general class of pure non-Gaussian states is composed of those with corresponding continuous-variable core states:

$$|\psi(\lambda)\rangle = \hat{U}(\theta)\hat{D}(\alpha)\hat{S}(r)\left(\sum_{n=0}^{\infty} c_n(\lambda)|n\rangle\right). \tag{5}$$

In contrast to the previous class, such targets cannot be perfectly prepared by either photon-subtraction or -addition techniques as they would require infinite non-Gaussian resources. Of course, these states are typically generated from a suitable Hamiltonian by a time evolution. Consequently, such states can only be prepared approximately as

$$|\psi_C^N(\lambda)\rangle = \hat{U}(\theta)\hat{D}(\alpha)\hat{S}(r)\left(\frac{\sum_{n=0}^{N} c_n(\lambda)|n\rangle}{\sqrt{\sum_{n=0}^{N}|c_n(\lambda)|^2}}\right) \tag{6}$$

to an accuracy determined by the fidelity with the target:

$$\mathcal{F}_C = |\langle\psi(\lambda)|\psi_C^N(\lambda)\rangle|^2. \tag{7}$$

A little algebra reveals that the fidelity is a function of the Gaussian parameters (r, α, θ) and the non-Gaussian resource cost N. This follows since the truncated core is given by

$$|\lambda, r, \alpha, \theta; N\rangle = \frac{\hat{\Pi}_N|\lambda, r, \alpha, \theta\rangle}{\sqrt{\langle\lambda, r, \alpha, \theta|\hat{\Pi}_N|\lambda, r, \alpha, \theta\rangle}}, \tag{8}$$

where $\hat{\Pi}_N = \sum_{n=0}^{N}|n\rangle\langle n|$ projects onto an N-dimensional subspace. Thus, the approximate target state from a truncated core is then

$$|\psi_C^N(\lambda)\rangle = \hat{U}(\theta)\hat{D}(\alpha)\hat{S}(r)|\lambda, r, \alpha, \theta; N\rangle, \tag{9}$$

and so

$$\langle\psi(\lambda)|\psi_C^N(\lambda)\rangle = \langle\lambda, r, \alpha, \theta|\lambda, r, \alpha, \theta; N\rangle \tag{10}$$

by virtue of unitarity of the Gaussian operations. Accordingly

$$\langle\psi(\lambda)|\psi_C^N(\lambda)\rangle = \frac{\langle\lambda, r, \alpha, \theta|\hat{\Pi}_N|\lambda, r, \alpha, \theta\rangle}{\sqrt{\langle\lambda, r, \alpha, \theta|\hat{\Pi}_N|\lambda, r, \alpha, \theta\rangle}}, \tag{11}$$

which leads to the conclusion

$$\mathcal{F}_C(\lambda, r, \alpha, \theta, N) = \langle\lambda, r, \alpha, \theta|\hat{\Pi}_N|\lambda, r, \alpha, \theta\rangle. \tag{12}$$

For this latter class of states, we first fix λ and N and then we optimize the phase θ, squeezing r, and displacement α to maximize the fidelity and, hence, obtain the optimal agreement between $|\psi(\lambda)\rangle$ and $|\psi_C^N(\lambda)\rangle$. This optimization process unifies several issues related to the state preparation of non-Gaussian pure states. First, it identifies the essential non-Gaussian operational cost that underlies each non-Gaussian pure state. Second, it highlights the possible trade-offs between Gaussian and non-Gaussian operations in this form of state preparation. In general, this optimization must be performed numerically due to the nontrivial nature of Fock-state decomposition for each core. This is given by

$$|\lambda,r,\alpha,\theta\rangle = \sum_{n,m,k=0}^{\infty} e^{-i\theta k} S_{nm}(-r) D_{mk}(-\alpha^*)\psi_k(\lambda)|n\rangle,$$

where the displacement matrix elements are given by [41] $\langle m|\hat{D}(\beta)|k\rangle = D_{mk}(\beta)$ with

$$D_{mk}(\beta) = (m!/k!)^{-1/2} e^{-|\beta|^2/2}(-\beta^*)^{k-m} L_m^{k-m}(|\beta|^2)$$

for $m \leqslant k$ and

$$D_{mk}(\beta) = (k!/m!)^{-1/2} e^{-|\beta|^2/2}\beta^{m-k} L_k^{m-k}(|\beta|^2) \tag{13}$$

for $m \geqslant k$. Note that the $L_k^{m-k}(|\beta|^2)$ are the generalized *Laguerre* polynomials. The matrix coefficients for the squeezing operator [42] are $\langle n|\hat{S}(r)|m\rangle = S_{nm}(r)$. When both m and n are even integers,

$$S_{nm}(r) = \frac{(-1)^{m/2}}{(m/2)!(n/2)!}\sqrt{\frac{n!m!}{\cosh r}}\left(\frac{\tanh r}{2}\right)^{(n+m)/2} \times {}_2F_1\left(-\frac{m}{2},-\frac{n}{2};\frac{1}{2};-\frac{1}{\sinh^2 r}\right), \tag{14}$$

but when both m and n are odd,

$$S_{nm}(r) = \frac{(-1)^{(m-1)/2}}{\left(\frac{m-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\sqrt{\frac{n!m!}{\cosh^3 r}}\left(\frac{\tanh r}{2}\right)^{(n+m)/2-1} \times {}_2F_1\left(-\frac{(m-1)}{2},-\frac{(n-1)}{2};\frac{3}{2};-\frac{1}{\sinh^2 r}\right), \tag{15}$$

and $S_{nm}(r)$ vanish for all other possibilities. Note that ${}_2F_1$ are Gauss hypergeometric polynomials [42].

B. Utility of unitary Gaussian operations

The main inspiration of this work was whether Gaussian unitary operations can reduce the non-Gaussian cost, with respect to photon subtraction and addition schemes, involved in preparing a desired non-Gaussian target. The extent to which this is true is revealed by comparing the non-Gaussian resources required to prepare a direct truncation of $|\psi(\lambda)\rangle$ with that required for the minimum core state. Essentially, we determine the utility of Gaussian operations by considering the approximate preparation of $|\psi(\lambda)\rangle$ with and without them. This can be done by comparing the fidelities of the states produced by each method. These fidelities are defined as

$$\mathcal{F}_{DT}(\lambda,N) = |\langle\psi(\lambda)|\psi_{DT}^N(\lambda)\rangle|^2, \tag{16}$$

with

$$\mathcal{F}_{DT}(\lambda,N) = \langle\psi(\lambda)|\hat{\Pi}_N|\psi(\lambda)\rangle = \sum_{n=0}^{N}|\psi_n(\lambda)|^2 \tag{17}$$

where $\psi_n(\lambda) = \langle n|\psi(\lambda)\rangle$, for the direct truncation method and

$$\mathcal{F}_C = \langle\lambda,r,\alpha,\theta|\hat{\Pi}_N|\lambda,r,\alpha,\theta\rangle = \sum_{n=0}^{N}|c_n|^2 \tag{18}$$

for the core-state method. The superiority of the core-state method can be established on two levels corresponding to the two questions asked in the Introduction. First, the core-state method is better than the direct truncation method using the same non-Gaussian resources if

$$\langle\lambda,r,\alpha,\theta|\hat{\Pi}_N|\lambda,r,\alpha,\theta\rangle > \langle\psi(\lambda)|\hat{\Pi}_N|\psi(\lambda)\rangle. \tag{19}$$

The second condition, if true, that would demonstrate the superiority of the core method using *less* non-Gaussian resources over the direct truncation method is

$$\langle\lambda,r,\alpha,\theta|\hat{\Pi}_M|\lambda,r,\alpha,\theta\rangle \geqslant \langle\psi(\lambda)|\hat{\Pi}_N|\psi(\lambda)\rangle \tag{20}$$

for $M < N$. That is, we would expect that one can better the fidelity with the target by our approach with potentially *less* non-Gaussian resources than simply building a truncated target.

For the first class of states with finite-dimensional cores, this is obviously true. This is because we can perfectly prepare the state by preparing the core first and then applying unitary Gaussian operations

$$\mathcal{F}_C(N) = \sum_{n=0}^{N}|c_n|^2 = 1. \tag{21}$$

In contrast, the direct truncation method yields

$$\mathcal{F}_{DT}(N) = \sum_{n=0}^{N}|\psi_n(\lambda)| \leqslant 1, \tag{22}$$

because $|\psi(\lambda)\rangle$ is, in general, an infinite-dimensional state and is only reproduced with unity fidelity as $N \to \infty$ and so

$$\lim_{N\to\infty}[\mathcal{F}_{DT}(N)] = \mathcal{F}_C(N) = 1. \tag{23}$$

Thus, we would need infinite non-Gaussian resources to perfectly prepare the target by the direct truncation method. The reason for this is because in the direct truncation method the non-Gaussian subtractions and additions also contribute to building the Gaussian envelope of the state in addition to its non-Gaussian core. In contrast, in the core method all of the non-Gaussian resources are concentrated to preparing the non-Gaussian part of the state. Thus, for the example of the photon added coherent state (4), the core method is superior since none of the non-Gaussian subtractions and additions contribute to the construction of the displacement operator. In contrast, in the direct truncation method, each subtraction or addition contributes to building both the core and the displacement operator.

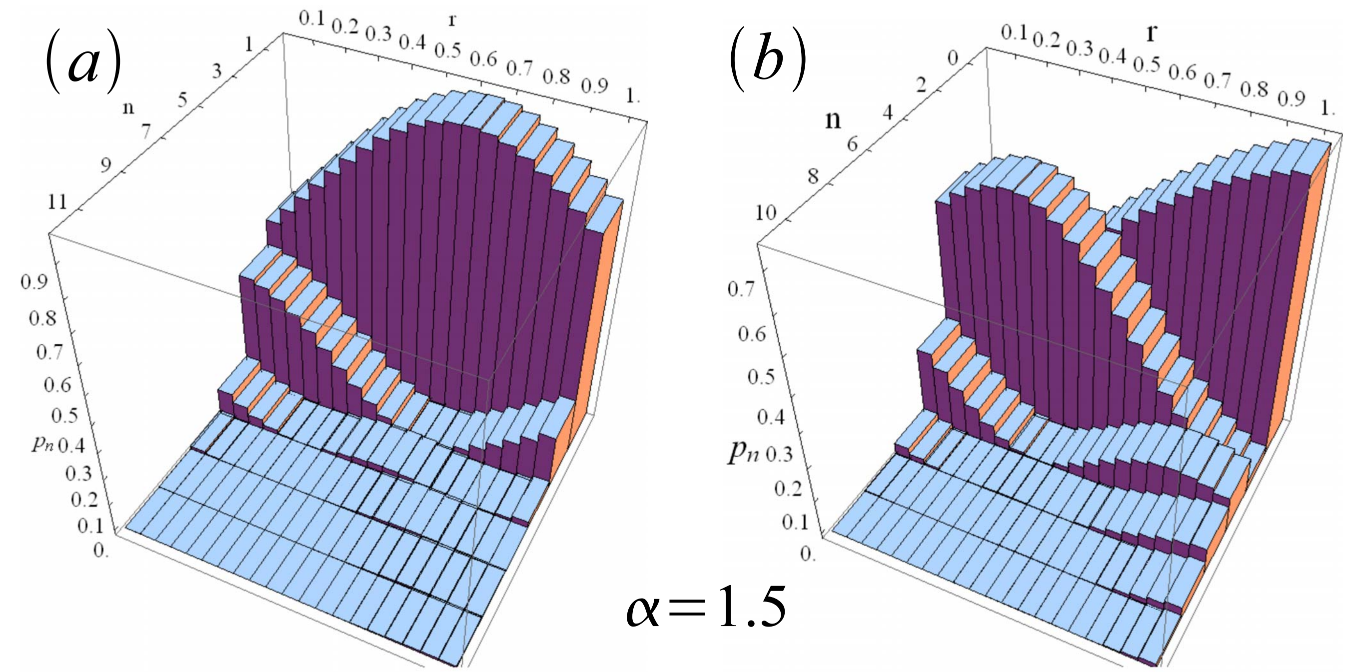

FIG. 1. (Color online) Each core state $|\alpha=1.5,r\rangle$ is labeled by a different squeezing and possesses a diverse range of photon-number probability distributions. This is shown for (a) the odd-parity cat and (b) the even-parity cat.

For the second class of states with infinite-dimensional cores, the situation is more subtle since both converge to unit fidelity as $N\to\infty$. However, proving the optimal nature of the core-state method to a direct truncation method for arbitrary pure target states is a nontrivial task and will not be tackled here. Instead, we provide some examples of states for which the core method is indeed superior to direct truncation. Specifically, in the next section, we consider the Schrödinger cat states and demonstrate that they support our case.

III. EXAMPLE: SCHRÖDINGER CAT STATES

The odd-parity superposition of coherent states $|\psi(\alpha)\rangle = \mathcal{N}(|\alpha\rangle - |-\alpha\rangle)$ (where, for simplicity, we assume $\alpha\in\Re$) is a well-known non-Gaussian state and is the subject of numerous theoretical quantum information protocols [43,44]. The characteristic feature of this state, from a photon number point of view, is the exclusion of all even Fock states $|\psi(\alpha)\rangle = \mathcal{N}'\sum_{n=0}^{\infty}\alpha^{2n+1}/\sqrt{(2n+1)!}|2n+1\rangle$. Consequently, each core state $|\alpha,r,\beta,\theta\rangle$, where $\beta\in\Re$, has the Fock decomposition

$$|\alpha,r,\beta,\theta\rangle \propto \sum_{n,m,k=0}^{\infty}(\alpha e^{-i\theta})^{2k+1} \times\left(\frac{S_{2n,2m}(-r)D_{2m,2k+1}(-\beta)}{\sqrt{(2k+1)!}}|2n\rangle + \frac{S_{2n+1,2m+1}(-r)D_{2m+1,2k+1}(-\beta)}{\sqrt{(2k+1)!}}|2n+1\rangle\right).$$

Thus, displacement acts to destroy the parity of the state since it destroys the symmetry of the state around the origin of phase space. Consequently, there is good reason to regard the optimal displacement for the cat as zero. Moreover, the optimal phase is also zero since we assumed $\alpha\in\Re$. This is also evident from our numerical simulations, and so we will restrict our attention to core states related to the target via squeezing alone. Accordingly, the core states are of the form

$$|\alpha,r\rangle = \mathcal{N}'\sum_{n,k=0}^{\infty}\frac{\alpha^{2k+1}S_{2n+1,2k+1}(-r)}{\sqrt{(2k+1)!}}|2n+1\rangle, \tag{24}$$

where $\mathcal{N}'$ is a normalization factor, and so the photon-number probability distribution of a core state is a function of the squeezing parameter r. This behavior is readily illustrated in Fig. 1(a), where it can be observed that for $\alpha=1.5$, each core exhibits a different photon-number probability distribution. The most important point from this is that different cores will require a different number of minimum subtractions to approximately prepare the desired target state.

Each core state, when truncated, yields an approximation to the initial target state $|\psi(\alpha)\rangle$. These approximate target states are given by

$$|\psi_C^N(\alpha)\rangle = \mathcal{N}''\sum_{m,k=0}^{\infty}\frac{\alpha^{2k+1}A^N_{2m+1,2k+1}}{\sqrt{(2k+1)!}}|2m+1\rangle, \tag{25}$$

where $A^N_{2m+1,2k+1}=\sum_{n=0}^{N}S_{2m+1,2n+1}(r)S_{2n+1,2k+1}(-r)$—i.e., $|\psi^N(\alpha)\rangle \propto \hat{S}(r)\hat{\Pi}_N|\alpha,r\rangle$ with $\mathcal{N}''$ as a normalization factor. The fidelity between the actual desired target state $|\psi(\alpha)\rangle$ and each of the approximate targets is then defined as

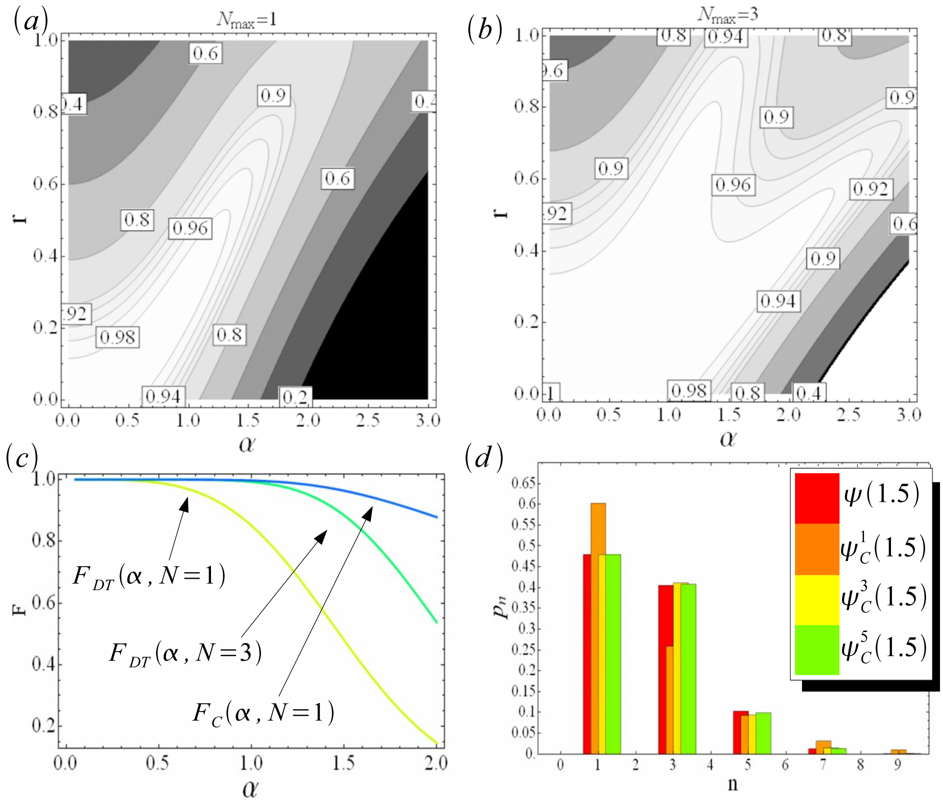

FIG. 2. (Color online) (a) displays the relation between α and r for different constant values of the fidelity $\mathcal{F}_C(\alpha,r,N=1)$ for one-photon subtraction. (b) shows the same information—i.e., $\mathcal{F}_C(\alpha,r,N=3)$. (c) displays a comparison of the fidelities between the target state and states from our core method and direct truncation. Finally, (d) compares the photon-number distributions of the approximate targets with the actual target $|\psi(\alpha)\rangle$ for $\alpha=1.5$ with $N=1,3,5$. The optimal squeezing for these cores is $r=(0.597,0.263,0.157)$, giving fidelities $\mathcal{F}_C=(0.9638,0.9995,0.9999)$.

$$\mathcal{F}_C(\alpha,r,N)=\sum_{n=0}^{N}\left|\mathcal{N}''\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}S_{2n+1,2k+1}(-r)}{\sqrt{(2k+1)!}}\right|^2, \qquad (26)$$

and the optimal core state for a given N is obtained by maximizing this quantity. This optimization is performed numerically due to its complexity, but we can still gain an insight into the relationship between α and r for constant values of the above fidelity. For example, when $\alpha=1.5$, numerical optimization of the fidelity yields $F=0.96$ for $r=0.597$ and $N=1$ and $F=0.999\,505$ for $r=0.263$ and $N=3$. This is precisely the content of Figs. 2(a) and 2(b), which show the relation between α and r for fixed non-Gaussian resources of $N=1$ and 3, respectively. In addition to this, it is important to show that the preparation of the target state $|\psi(\alpha)\rangle$ via the optimal core and Gaussian operations is more economical than a direct production of a truncated version of the target from non-Gaussian operations only.

To demonstrate that this is indeed the case, we show that the core method can produce a cat to an equal or better accuracy for a smaller number of photon subtractions. This is shown in Fig. 2(c), where we compare the fidelities $\langle\alpha,r|\hat{\Pi}_N|\alpha,r\rangle$ and $\langle\psi(\alpha)|\hat{\Pi}_M|\psi(\alpha)\rangle$ for $N<M$ and reveal the instances where our core preparation method is more economical. In particular, we note that the fidelity using the core with $N=1$ is better than that of the truncated version of the target for both $M=1$ and 3. Thus, instead of attempting to successfully perform three successive subtractions to approximately prepare $|\psi(\alpha)\rangle$ for $0<\alpha<2$, we need only perform a single subtraction and then squeeze the state accordingly.

Finally, it is important to compare the photon-number distributions of the approximate targets $(|\psi_C^1(\alpha)\rangle,|\psi_C^3(\alpha)\rangle,|\psi_C^5(\alpha)\rangle)$ with the actual target $|\psi(\alpha)\rangle$. This is illustrated in Fig. 2(d) for $\alpha=1.5$, where it is clear that the latter two approximate targets provide an excellent approximation to $|\psi(1.5)\rangle$. Thus, for $\alpha=1.5$, the squeezed single-photon state as the core lacks a sufficient accuracy. This fact is also readily evident when one consults the contour plot in Fig. 2(a) as there are no contours that satisfy $\mathcal{F}_C(1.5,r,1)\geqslant 0.96$. This particular example provides a concrete understanding of our proposal and illustrates the main features of it.

An identical analysis can be performed on the even-parity cat state $|\phi(\alpha)\rangle=\mathcal{M}(|\alpha\rangle+|-\alpha\rangle)$ with $\alpha\in\mathfrak{R}$. In this case, we find that all the essential points of the previous example are repeated. First, the even symmetry of this state, $|\phi(\alpha)\rangle\propto\sum_{n=0}^{\infty}\alpha^{2n}/(\sqrt{(2n)!})|2n\rangle$, means that the optimal displacement and phase are both zero. Consequently, each core state is labeled by the corrective squeezing

$$|\alpha,r\rangle=\mathcal{M}'\sum_{k,n=0}^{\infty}\frac{S_{2n,2k}(-r)\alpha^{2k}}{\cosh\alpha^2(2k)!}|2n\rangle \qquad (27)$$

and has a different photon-number decomposition as shown in Fig. 1(b). Again, the reason that the external squeezing is useful is because it preserves the symmetry of the state. Furthermore, each core state, when truncated, yields an approximation to the even-parity cat. These approximate target states are given by

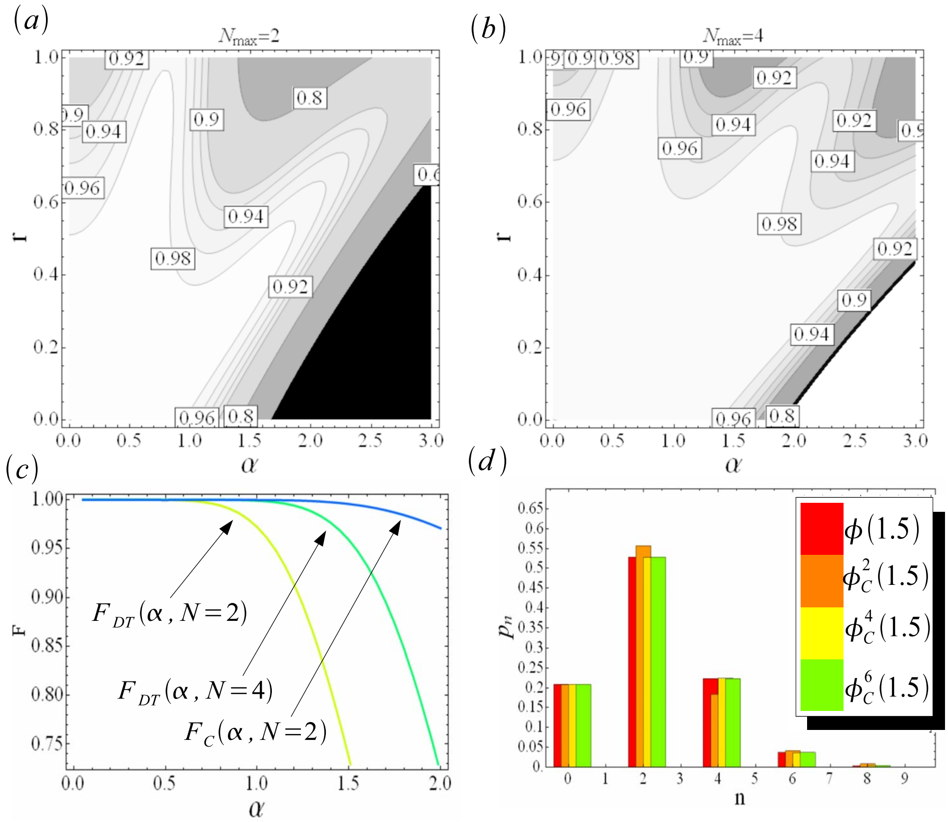

FIG. 3. (Color online) (a) displays the relation between α and r for different constant values of the fidelity $\mathcal{F}_C(\alpha,r,N=2)$ for two-photon subtractions. (b) shows the same information—i.e., $\mathcal{F}_C(\alpha,r,N=4)$. (c) displays a comparison of the fidelities between the target state and states from our core method and direct truncation. Finally, (d) compares the photon number distributions of the approximate targets with the actual target $|\phi(\alpha)\rangle$ for $\alpha=1.5$ with $N=2,4,6$. The optimal squeezing for these cores are $r=(0.376,0.198,0.272)$, giving fidelities $\mathcal{F}_C=(0.9958,0.9999,0.9999)$.

$$|\phi_C^N(\alpha)\rangle = \mathcal{M}'' \sum_{m,k=0}^{\infty} \frac{\alpha^{2k} B_{2m,2k}^N}{\sqrt{(2k!)}} |2m\rangle, \tag{28}$$

where $B_{2m,2k}^N = \sum_{n=0}^N S_{2m,2n}(r) S_{2n,2k}(-r)$ and $\mathcal{M}''$ is a normalization factor. The fidelity between the actual desired target state $|\psi(\alpha)\rangle$ and each of the approximate targets is then defined as

$$\mathcal{F}_C(\alpha,r,N) = \sum_{n=0}^{N} \left| \mathcal{M}'' \sum_{k=0}^{\infty} \frac{\alpha^{2k} S_{2n,2k}(-r)}{\sqrt{(2k)!}} \right|^2, \tag{29}$$

and the optimal core state for a given N is obtained by maximizing this quantity. Just as in the previous case for the odd-parity cat, we consider this fidelity for $\alpha=1.5$ in the case of $N=2$ and $N=4$. Already, one can consider the preparation of the even-parity cat state more complicated than the odd one since the most basic even cat will require two-photon subtractions rather than one.

All of this is shown in Fig. 3. First, in Figs. 3(a) and 3(b), we plot the $\mathcal{F}_C(\alpha,r,N)$ for $N=2,4$ where we once again see the nontrivial and nonunique relationship between α and the optimal squeezing. In Fig. 3(c), we once again demonstrate that employing Gaussian operations is an advantage since it allows an improvement in the accuracy of approximating $|\phi(\alpha)\rangle$ for fewer photon subtractions than required for the direct truncation method. Finally, Fig. 3(d) shows how the states prepared by the core method approximate the desired target $|\phi(\alpha=1.5)\rangle$. These examples are particularly elegant due to their inherent symmetry. In principle, this method could provide key insights into other desirable non-Gaussian pure states and their approximate preparation.

IV. DISCUSSION AND CONCLUDING REMARKS

In summary, we have proposed that Gaussian operations can reduce the required non-Gaussian resources for pure-state preparation. It is a nontrivial problem to establish the exact nature of this trade-off and ascertain whether it applies to all non-Gaussian pure states. Instead, we are limited to analyzing the properties of each desired target non-Gaussian state to determine if Gaussian operations are advantageous. Unfortunately, being able to demonstrate that this is true in general for arbitrary pure non-Gaussian states is a nontrivial task.

It remains an open question as to the application of this method to mixed states [45–47], which is likely to be a challenging problem. An insight into this can be gained by considering the attenuated version of the state $|\psi(\alpha)\rangle$ as the target state characterized by a transmission η. The target can be written as $p|\psi(\eta\alpha)\rangle\langle\psi(\eta\alpha)|+(1-p)/2(|\eta\alpha\rangle\langle\eta\alpha|+|-\eta\alpha\rangle\langle-\eta\alpha|)$, where $p=p(\alpha,\eta)$. Thus, it is enough to prepare the pure state $|\psi(\eta\alpha)\rangle$ and obtain the target by applying additional random operations which add the mixture of two coherent states. It follows that, to find the core state, we have to subtract the non-Gaussian noise contribution from the target state. However, identifying this non-Gaussian noise remains an open problem. On the other hand, the majority of the desired non-Gaussian states are pure; therefore, our result is sufficient for all practical purposes. Another outstanding issue raised by our work is the assumption of perfect photon-subtraction techniques. If we relax this assumption to consider noisy detectors, then both the number of photon subtractions and the purity of each implemented subtraction could be advanced as a cost function for state preparation. This generalization would be an interesting problem to pursue. Moreover, our work here provides motivation to further investigate the potential benefits of Gaussian operations on manipulations of non-Gaussian states including transmission through noisy channels, measurement-induced nonlinearity schemes, and the preparation of non-Gaussian entangled states. In this way, we will come closer to understanding the subtle interplay between the Gaussian and non-Gaussian structure of nonclassical resources in quantum information.

ACKNOWLEDGMENTS

This work was supported by the Engineering and Physical Sciences Research Council and as part of the following projects: No. MSM 6198959213 of MSMT CR, No. 202/08/0224 of GA CR, the Alexander von Humboldt Foundation, and EU Grant No. FP7 212008COMPAS.

[1] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).
[2] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).
[3] R. Short and L. Mandel, Phys. Rev. Lett. **51**, 384 (1983).
[4] A. I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, Phys. Rev. Lett. **87**, 050402 (2001).
[5] A. Ourjoumtsev, R. Tualle-Brouri, and Ph. Grangier, Phys. Rev. Lett. **96**, 213601 (2006).
[6] A. I. Lvovsky and J. Mlynek, Phys. Rev. Lett. **88**, 250401 (2002).
[7] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and Ph. Grangier, Science **312**, 83 (2006).
[8] J. Wenger, R. Tualle-Brouri, and Ph. Grangier, Phys. Rev. Lett. **92**, 153601 (2004).
[9] S. Olivares and M. G. A. Paris, J. Opt. B: Quantum Semiclassical Opt. **7**, S616 (2005).
[10] J. S. Neergaard-Nielsen, B. Melholt Nielsen, C. Hettich, K. Molmer, and E. S. Polzik, Phys. Rev. Lett. **97**, 083604 (2006).
[11] M. Dakna, T. Anhut, T. Opatrný, L. Knöll, and D. G. Welsch, Phys. Rev. A **55**, 3184 (1997).
[12] A. Zavatta, S. Viciani, and M. Bellini, Science **306**, 660 (2004).
[13] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouri, and Ph. Grangier, Nature (London) **448**, 784 (2007).
[14] T. Opatrný, G. Kurizki, and D. G. Welsch, Phys. Rev. A **61**, 032302 (2000).
[15] S. Olivares, M. G. A. Paris, and R. Bonifacio, Phys. Rev. A **67**, 032314 (2003).
[16] F. Dell'Anno, S. De Siena, L. Albano, and F. Illuminati, Phys. Rev. A **76**, 022301 (2007).
[17] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles, Phys. Rev. A **73**, 042310 (2006).
[18] C. Invernizzi, S. Olivares, M. G. A. Paris, and K. Banaszek, Phys. Rev. A **72**, 042105 (2005).
[19] S. Olivares and M. G. A. Paris, J. Opt. B: Quantum Semiclassical Opt. **7**, S392 (2005).
[20] R. García-Patrón, J. Fiurášek, N. J. Cerf, J. Wenger, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **93**, 130409 (2004).
[21] S. Olivares and M. G. A. Paris, Phys. Rev. A **70**, 032112 (2004).
[22] R. García-Patrón, J. Fiurášek, and N. J. Cerf, Phys. Rev. A **71**, 022105 (2005).
[23] H. Jeong, Phys. Rev. A **78**, 042101 (2008).
[24] A. Ourjoumtsev, A. Dantan, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **98**, 030502 (2007).
[25] M. Takeoka and M. Sasaki, Phys. Rev. A **75**, 064302 (2007).
[26] J. Fiurášek, R. García-Patrón, and N. J. Cerf, Phys. Rev. A **72**, 033822 (2005).
[27] M. Dakna, J. Clausen, L. Knöll, and D. G. Welsch, Phys. Rev. A **59**, 1658 (1999).
[28] A. E. B. Nielsen and K. Mölmer, Phys. Rev. A **76**, 043840 (2007).
[29] M. Sasaki, M. Takeoka, and H. Takahashi, Phys. Rev. A **77**, 063840 (2008).
[30] M. Takeoka, H. Takahashi, and M. Sasaki, Phys. Rev. A **77**, 062315 (2008).
[31] U. Titulaer and R. Glauber, Phys. Rev. **140**, B676 (1965).
[32] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[33] L. Mandel, Phys. Scr. **T12**, 34 (1986).
[34] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000).
[35] M. S. Kim, W. Son, V. Buzek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002).
[36] J. K. Asboth, J. Calsamiglia, and H. Ritsch, Phys. Rev. Lett. **94**, 173602 (2005).
[37] E. Shchukin, T. Richter, and W. Vogel, Phys. Rev. A **71**, 011802(R) (2005).
[38] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Phys. Rev. A **76**, 042327 (2007).
[39] M. G. Genoni, M. G. A. Paris, and K. Banaszek, e-print arXiv:0805.1645.
[40] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, England, 1997).
[41] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Continuous Variable Quantum Information* (Bibliopolis, Napoli, 2005).
[42] P. Marian, Phys. Rev. A **44**, 3325 (1991).
[43] T. C. Ralph, Rep. Prog. Phys. **69**, 853 (2006).
[44] *Quantum Information with Continuous Variables of Atoms and Light*, edited by N. J. Cerf, G. Leuchs, and E. S. Polzik (Imperial College Press, London, 2007).
[45] A. Zavatta, V. Parigi, and M. Bellini, Phys. Rev. A **75**, 052106 (2007).
[46] M. S. Kim, E. Park, P. L. Knight, and H. Jeong, Phys. Rev. A **71**, 043805 (2005).
[47] K. Molmer, Phys. Rev. A **73**, 063804 (2006).