Quantum-controlled measurement device for quantum-state discrimination

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We propose a “programmable” quantum device that is able to perform a specific generalized measurement from a certain set of measurements depending on a quantum state of a “program register.” In particular, we study a situation when the programmable measurement device serves for the unambiguous discrimination between nonorthogonal states. The particular pair of states that can be unambiguously discriminated is specified by the state of a program qubit. The probability of successful discrimination is not optimal for all admissible pairs. However, for some subsets it can be very close to the optimal value.

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I. INTRODUCTION

Quantum measurements are inevitable parts of all quantum devices. Specific sets of quantum measurements are essential for optimal quantum-state estimation [1,2]. Quantum measurements also represent the final step of any quantum computation [3]. In many situations the choice of an optimal measurement depends on the task to be performed. For instance, in the case of quantum-state discrimination the choice of the measurement is given by the specific pair of states that are supposed to be discriminated. A selection of a specific measurement can be performed on a “classical” level. That is, the parameters of the measurement (e.g., the orientation of the Stern-Gerlach apparatus) are completely described classically. On the other hand the parameters determining the character of quantum measurement can be encoded in a state of a quantum “program” register. Certainly, in this situation one could perform a measurement on a program register and estimate the parameters specifying the measurement to be performed on the system. With these parameters one can then “classically” adjust the measurement apparatus and perform the measurement over the system. The other option is that the quantum program register directly determines the measurement to be performed on the system. This purely quantum control can be realized without an intermediate intervention of an observer.

Therefore it is interesting to understand whether it is possible to construct a universal (multipurpose) quantum-measurement device (“quantum multimeter”). That is, an apparatus that could perform a specific class of generalized measurements [positive operator valued measure (POVM)] in such a way that each member of this class could be selected by a particular quantum state of a “program register.” The key property of this approach is a possibility to control the choice of the measurement (e.g., the measurement basis in case of a projective measurement) by a (in principle, unknown) quantum state of the program register. This state can be determined, for instance, as a result of some quantum-information process.

The generalized measurement is defined by the fact that the probability of each of its results (the number of results may be, in general, larger than the dimension of the Hilbert space of the measured system) is given by the expression $p_\mu = \text{Tr}(A_\mu \rho_S)$, where $\rho_S$ is the state of the system and $A_\mu$ are positive operators that constitute the decomposition of the identity operator ($\sum_\mu A_\mu = 1$). This is the reason why it is called positive operator valued measure [1,2,4]. Each POVM can be implemented using an ancillary quantum system in a specific state and realizing a projective von Neumann measurement on the composite system [5]. In other words, if one has an “input” (measured) state $\rho_S$ in the Hilbert space $\mathcal{H}_S$ it is always possible to find some state $\rho_A$ in a space $\mathcal{H}_A$ and a set of orthogonal projectors $\{E_\mu\}$ acting on $\mathcal{H}_S \otimes \mathcal{H}_A$ ($\sum_\mu E_\mu = 1$) such that

$$A_\mu = \text{Tr}_A(E_\mu \rho_A)$$

are positive operators as discussed above.

In general, we can assume that the initial state of the ancilla can be prepared with an arbitrary precision. The ancilla can be considered as a part of the “program register.” Further, we note that the general projection measurement on the composite system can be represented by a unitary transformation on the composite system followed by a fixed projection measurement (e.g., independent projective measurements on individual qubits). Therefore the problem of designing the programmable quantum multimeter reduces to the question of whether an arbitrary unitary operation (on the Hilbert space with a given dimension) can be encoded in some quantum state of a program register of a finite dimension. It was shown that the answer to this question is “No.” Nielsen and Chuang proved that any two inequivalent operations require orthogonal program states [6]. Thus the number of encoded operations cannot be higher than the dimension of the Hilbert space of the program register. Since, in general, the set of all unitary operations can be infinite, the result of Nielsen and Chuang implies that no universal programmable gate array can be constructed using finite resources. They showed, however, that if the gate array is probabilistic, a universal gate array is possible. A probabilistic array is one that requires a measurement to be made at the output of the program register, and the output of the data register is only accepted if a particular result, or set of results, is obtained. This will happen with a probability, which is less than one.
Vidal and Cirac [7] have presented a probabilistic program-
mable quantum gate array with a finite program register
which can realize a family of operations with one continuous
parameter. Recently, Hillery et al. [8] have proposed a more
general quantum processor that can perform probabilistically
any operation (not only unitary) on a qubit. Another aspect of
encoding quantum operations in states of a program register
has been discussed by Huelga et al. [9]. They dealt with the
so-called teleportation of unitary operations. Unfortunately,
the probabilistic realization of unitary operations cannot help
to build a programmable quantum multimeter in the way
mentioned above. The reason is that the probabilistic imple-
mentation of a given operation leads, at the end, to a different
POVM than the deterministic implementation of the same
operation would lead to. (The new \( N + 1 \) component POVM
with one more output corresponding to a “failure” is differ-
ent from the desired \( N \) component one. For example, if the
desired POVM already contains an inconclusive output then
if it is implemented probabilistically the total probability of
the “failure” increases in general.)

In general, we can describe a quantum multimeter as a
(fixed) unitary operation acting on the measured system (or a
“data register”) and an ancillary system (“program regis-
ter”) together and a (fixed) projective measurement realized
afterwards on the same composite system. Clearly, such a
device can perform only a restricted set of POVMs. One can,
therefore, ask what is the optimal unitary transformation that
enables us to implement “the largest set of POVMs” (in
comparison with the set of POVMs that would be obtainable
when we allowed any unitary transformation on the same
Hilbert space). One can also ask what unitary transformation can
help to approximate all the POVMs (generated by an
arbitrary unitary transformation) with the highest precision
(fidelity) on average. Clearly, the last task requires definition
of the distance measure between two POVMs. This is an
interesting problem per se, however, it goes far beyond the
scope of our considerations here. Both optimization prob-
lems mentioned above are rather nontrivial. Moreover, the
introduced scheme is perhaps too general from a practical
point of view. Therefore in the present paper we will concen-
trate our attention on a more specific case: On the problem of
state discrimination.

We stress once again that a quantum multimeter as dis-
cussed in the present paper is a device which, in contrast to
its classical counterpart, is controlled (switched, pro-
grammed) by the quantum states of a program register that
are allowed to be mutually nonorthogonal.

II. DISCRIMINATION OF QUANTUM STATES

In the following we will study a particular example of a
“quantum multimeter” serving for a programmable unam-
biguous state discrimination. So, it is in place to say a few
words about quantum-state discrimination now.

A general unknown quantum state cannot be determined
completely by a measurement performed on a single copy of
the system. But the situation is different if a priori knowl-
edge is available [1,2,4], e.g., if one works only with states
from a certain discrete set. Even quantum states that are mu-
truly nonorthogonal can be distinguished with a certain
probability provided they are linearly independent (for a re-
view see Ref. [10]). There are, in fact, two different optimal
strategies [11]: First, the strategy that determines the state
with the minimum probability for the error [1,2] and, second,
unambiguous or error-free discrimination (the measurement
result never wrongly identifies a state) that allows the possi-
bility of an inconclusive result (with a minimal probability in
the optimal case) [12–16]. We will concentrate our attention
to the unambiguous state discrimination. It has been first
investigated by Ivanovic [12] for the case of two equally
probable nonorthogonal states. Peres [14] solved the problem
of discrimination of two states in a formulation with POVM
measurement. Later Jaeger and Shimony [15] extended the
solution to arbitrary a priori probabilities. Chefles and Bar-
nett [16] have generalized Peres’s solution to an arbitrary
number of equally probable states which are related by a
symmetry transformation. Unambiguous state discrimination
was already realized experimentally. The first experiment,
designed for the discrimination of two linearly polarized
states of light, was done by Huttner et al. [17]. There are also
some newer proposals of optical implementations [18]. The
interest in the quantum state discrimination is not only “acade-
ic,” unambiguous state discrimination can be used, e.g.,
as an efficient attack in quantum cryptography [19].

III. “UNIVERSAL” DISCRIMINATOR

Let us suppose that we want to discriminate unambigu-
ously between two known nonorthogonal states. However,
we would like to have a possibility to “switch” the apparatus
in order to be able to work with several different pairs of
states.

Let us have two (nonorthogonal) input states of a qubit.
We can always choose such a basis that they read \( \alpha_0 \left| 0_D \right> \pm \beta_0 \left| 1_D \right> \) with \( \alpha_0 = \cos(\varphi/2) \) and \( \beta_0 = \sin(\varphi/2) \); the value of
\( \varphi_0 \) can be from 0 to \( \pi/2 \) (\( \varphi_0 \) is the angle between the two
states). Let us have one additional ancillary qubit, initially in
a state \( \left| 0_A \right> \). On both the “data” and the ancilla we apply the
following unitary transformation \( U_{D0A} \):

\[
\begin{align*}
|0_D0_A\rangle &\rightarrow \cos \theta |0_D0_A\rangle + \sin \theta |0_D1_A\rangle, \\
|1_D0_A\rangle &\rightarrow |1_D0_A\rangle, \\
|0_D1_A\rangle &\rightarrow - \sin \theta |0_D0_A\rangle + \cos \theta |0_D1_A\rangle, \\
|1_D1_A\rangle &\rightarrow |1_D1_A\rangle,
\end{align*}
\]

(2)

where \( \cos \theta = \tan(\varphi_0/2) \). If we then make a von Neumann
measurement consisting of the projectors \( P_+ = |+\rangle\langle +|, P_- = |\rangle\langle \rangle \), and \( P_0 = 1 - P_+ - P_- \), where

\[
|\pm\rangle = (|0_D0_A\rangle \pm |1_D0_A\rangle)/\sqrt{2},
\]

we can unambiguously determine the input state (with a cer-
tain probability of success) [17]. This measurement is opti-
mal in the sense that the probability of an inconclusive result
is the lowest possible (and it is the same for both states). The probability of the successful discrimination is $2 \sin^2(\varphi/2)$ [14].

Let us suppose now the set of pairs

$$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle,$$
$$|\psi_2\rangle = \alpha|0\rangle - \beta|1\rangle,$$  \hspace{1cm} (4)

where $\alpha = \cos(\varphi/2)$ and $\beta = \sin(\varphi/2)$, for all $\varphi$ from the interval $(0, \pi)$. That is, we consider all pairs of states that lie on a real plane and that are located symmetrically around the state $|0\rangle$; see Fig. 1. Further, let us suppose that the ancillary qubit is allowed to be in an arbitrary pure state

$$|\Xi\rangle_\lambda = a|0\rangle + b|1\rangle.$$  \hspace{1cm} (5)

Thus the total input state reads

$$|\Psi\rangle_{DA} = (a|0\rangle \pm \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) = a|0\rangle \otimes \alpha|0\rangle + a|0\rangle \otimes \beta|1\rangle + b|1\rangle \otimes \alpha|0\rangle + b|1\rangle \otimes \beta|1\rangle.$$  \hspace{1cm} (6)

After the action of transformation (2) on this state one obtains the resulting state in the following form [the transformation is fixed for all $\psi$; still $\cos \theta = \tan(\varphi/2)$]

$$U_{DA}|\Psi\rangle_{DA} = (a a \cos \theta - ab \sin \theta)|0\rangle \otimes |0\rangle + (a a \sin \theta + ab \cos \theta)|0\rangle \otimes |1\rangle,$$
$$+ a b|1\rangle \otimes \alpha|0\rangle \pm \beta|1\rangle \otimes \beta|1\rangle.$$  \hspace{1cm} (7)

If the coefficients $a$ and $b$ in the state of the ancilla are chosen in such a way that

$$(a a \cos \theta - ab \sin \theta) = \beta a = q/\sqrt{2},$$  \hspace{1cm} (8)

then the expression (7) simplifies to the form

$$U_{DA}|\Psi\rangle_{DA} = q|\pm\rangle + \text{const}_{0}|0\rangle \otimes \pm \text{const}_{1}|1\rangle.$$  \hspace{1cm} (9)

where the states $|\pm\rangle$ are defined by Eq. (3). Clearly, applying the projective measurement introduced above one is able to discriminate unambiguously states (4) for any given $\varphi \in (0, \pi)$ provided he/she has prepared the proper state of the ancilla. The first term in Eq. (9) corresponds to the successful discrimination, while the last two terms correspond to inconclusive results. The probability of success is

$$P_{\text{succ}} = |q|^2 = P_{\text{suc}}R(\varphi, \varphi_0) = 2 \sin^2(\varphi/2) R(\varphi, \varphi_0),$$  \hspace{1cm} (10)

where

$$R(\varphi, \varphi_0) = \frac{\cos \varphi_0 (\cos \varphi + 1)}{1 + \cos \varphi_0 - \sin \varphi \sin \varphi_0}.$$  \hspace{1cm} (11)

is the ratio between the actual value of the probability of successful discrimination and its optimal value. This expression is obtained from the condition (8) together with the normalization relation $|a|^2 + |b|^2 = 1$.

From above it follows that it is possible to implement a "universal quantum multimeter" that is able to discriminate probabilistically but unambiguously (with no errors) between two nonorthogonal states for the large class of nonorthogonal pairs. The selection of the desired regime (i.e., the selection of the pair of states that should be unambiguously discriminate) is done by the choice of the quantum state of the ancillary qubit. This program state selects the measurement to be performed on the system. The probability of the successful discrimination can be optimal only for one such pair of states.

In the limit case when $\varphi_0 = 0$, i.e., $\theta = \pi/2$ (this is the fixed parameter of the employed unitary transformation), the probability of the successful discrimination for different $\varphi$'s (i.e., for different settings of the ancilla and different pairs of input states) is the same as in the "quasi-classical" case, $P_{\text{suc}} = 1/2 \sin^2 \varphi$. By a quasiclassical approach we mean the probabilistic measurement when one randomly selects [20] the projective measurement in one of two orthogonal basis that both span the two-dimensional space containing both nonorthogonal states of interest (4). One basis consists of the states $|\psi_1\rangle$ and its orthogonal complement $|\psi_2\rangle$. If one finds the result corresponding to $|\psi_1\rangle$, he/she can be sure that the state $|\psi_1\rangle$ was not present. Analogously, the other basis consists of the state $|\psi_2\rangle$ and its orthogonal complement.

On the other hand when $\varphi_0 = \pi/2$, i.e., $\theta = 0$, there is no way to fulfill the condition (8) with $a \neq 0$ (and $P_{\text{suc}} \neq 0$) unless $a = \beta = 1/\sqrt{2}$. That is, only two orthogonal states (3) can be unambiguously discriminated.

If the parameter $\varphi_0$ is somewhere in between 0 and $\pi/2$ the probability of success (as a function of $\varphi$) is very close to the optimal value in the relatively large vicinity of $\varphi_0$; see Fig. 2. However, for small values of $\varphi$ it goes below the success probability of the quasiclassical case and for $\varphi = \pi/2$ (orthogonal states) the probability of successful discrimination is lower than unity.

One can ask for the optimal value of $\varphi_0$ in the sense that the average probability of successful discrimination [or, alternatively, function $R(\varphi, \varphi_0)$] over some chosen interval of $\varphi$'s is maximal. For example, if we are interested in the average value of $R(\varphi, \varphi_0)$ over the interval of $\varphi$ from 0 to $\pi/2$ we find that it is maximized when $\varphi_0 = 0.235 \pi$ (the corresponding average value of $R$ is 0.92).
The probability of the successful discrimination of states where Re(\(\psi_0\)) denotes the real part of \(\psi_0\). The device can be set to discriminate unambiguously orthogonal states of qubit that works with a large set of pairs of measurement devices. However, it should be stressed that the particular pair of states that can be unambiguously discriminated is specified by the state of a “program” qubit. Two possible input states of the “data qubit” that are in correspondence with the program setting are never wrongly identified but from time to time we can get an inconclusive result. The probability of successful discrimination is optimal only for one program setting. However, the device can be designed in such a way that the probability of successful discrimination is very close to the optimal value for a relatively large set of program settings. Let us stress the quantum nature of the “programming.” The states of the program register that represent different programs can be nonorthogonal.

We have also discussed some general questions concerning the possibilities to build multipurpose quantum measurement devices (“quantum multimeters”) that could perform a required POVM depending on a quantum state of their program register. Most of these questions remain unanswered. For instance, let us suppose a set of all POVMs that can be obtained if we combine the measured system with an ancilla of some fixed dimension in an arbitrary state and carry out an arbitrary projective (von Neumann) measurement on the composite system. This is equivalent to carrying out an arbitrary unitary operation followed by some fixed projective measurement. Imagine now that we can change only the state of the ancilla but our projective measurement (or unitary transformation) is fixed. The question is: What measurement (operation) do we need to approximate all the POVMs from the set introduced above with the maximal average fidelity? Apparently, this question raises the other interesting task: How to define the distance between two POVMs? Such problems are not trivial, however, they open perspectives in investigation of programmable quantum devices.

IV. CONCLUSIONS

We have proposed a programmable quantum measurement device for the error-free discrimination of two nonorthogonal states of qubit that works with a large set of pairs of states. The device can be set to discriminate unambiguously any two states that are symmetrically located around some fixed state [in the sense of Eq. (4)]. The setting is done through the state of a program register that is represented by another qubit. This means that the particular pair of states that can be unambiguously discriminated is specified by the state of a “program” qubit. Two possible input states of the “data qubit” that are in correspondence with the program setting are never wrongly identified but from time to time we can get an inconclusive result. The probability of successful discrimination is optimal only for one program setting. However, the device can be designed in such a way that the probability of successful discrimination is very close to the optimal value for a relatively large set of program settings. Let us stress the quantum nature of the “programming.” The states of the program register that represent different programs can be nonorthogonal.

For pedagogical reasons, until now we have only worked with the states from a particular real subspace of the Hilbert space of the data qubit. However, it should be stressed that the method works for any two “input” states that are symmetrically displaced with respect to \(|\psi_0\rangle\). In other words, the condition (8) can be fulfilled for any complex \(\alpha\) and \(\beta\). Simply,

\[
\frac{b}{a} = \frac{1}{\sin \theta} \left( \cos \theta - \frac{\beta}{\alpha} \right). 
\]

The probability of the successful discrimination of states then reads

\[
P_{\text{succ}} = \frac{2 \sin \theta |\alpha \beta|^2}{1 - 2 \cos \theta \text{Re}(\alpha \beta)}, \tag{12}
\]

where \(\text{Re}(\alpha \beta)\) denotes the real part of \(\alpha \beta\).

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[20] With the same probabilities provided that the frequencies of the occurrence of the input states are also the same.