How Quantum Correlations Enhance Prediction of Complementary Measurements

Radim Filip, Miroslav Gavenda, Jan Soubusta, Antonín Černoch, and Miloslav Dušek

1Department of Optics, Palacký University, 17. listopadu 50, 772 00 Olomouc, Czech Republic
2Joint Laboratory of Optics of Palacký University and Institute of Physics of Academy of Sciences of the Czech Republic, 17. listopadu 50A, 772 00 Olomouc, Czech Republic

(Received 15 June 2004; published 27 October 2004)

Quantum correlations have attracted the attention of physicists since the early days of quantum mechanics. Einstein et al. used the features of quantum correlations in their argumentation against the completeness of quantum theory [1]. It was building on the fact that the result of any potential measurement on one subsystem of the properly chosen entangled pair can be predicted with certainty after the proper measurement on the other subsystem. Following this fact and a few “natural” assumptions they concluded that there must simultaneously exist “elements of reality” for two complementary observables. Later, it was shown that quantum mechanics predicts different values of certain correlations of measurement results than any local realistic theory. Inequalities, which have to be satisfied within the local realism, were derived by Bell [2]. The predictions of quantum mechanics have been convincingly experimentally confirmed using, e.g., pairs of photons entangled in polarizations [3]. In this Letter, we analyze quantitatively how the correlations between the qubits prepared in a general mixed state enhance our ability to predict the results of complementary projective measurements on one qubit when we know the measurement results on the other one. This enhancement can be described by the quantity that we will call complementary knowledge excess. We have derived a nontrivial bound on the knowledge excesses which is determined only by the maximal violation of Bell inequalities [4]. An experimental test of this restriction was performed using a two-photon Werner state prepared by means of spontaneous parametric down-conversion.

We assume a general mixed state \( \rho_{SM} \) of a “signal” qubit S and a “meter” qubit M. Performing two projective measurements \( \Pi_M, \Pi'_M \) on qubit M, the prediction of the results of mutually complementary measurements \( \Pi_S, \Pi'_S \) on qubit S can be improved. Complementarity of measurements on a qubit means that \( \text{Tr}\Pi'_{Sj}\Pi_{Sj} = \frac{1}{2} \) for any \( i, j = 0, 1 \) (\( \Pi_{Sj}, \Pi'_{Sj} \) being corresponding projectors). Assuming \( \rho_{S0} = |\Psi_S\rangle\langle \Psi_S|, \rho_{S1} = |\Psi'_S\rangle\langle \Psi'_S| \), we can expand the state \( \rho_{SM} \) in the form

\[
\rho_{SM} = w|\Psi_S\rangle\langle \Psi| \otimes \rho_M + w^\perp|\Psi'^S\rangle\langle \Psi'| \otimes \rho_M^\perp + \sqrt{w}u^\perp(|\Psi_S\rangle\langle \Psi| \otimes \chi_M + \text{h.c.}),
\]

where \( 0 \leq w, w^\perp \leq 1 \), \( w + w^\perp = 1 \) and the operator \( \rho_M, \rho_M^\perp, \chi_M \) depend on the choice of the measurement \( \Pi_S \). In order to predict the result of the measurement \( \Pi_S \) one needs to discriminate between the mixed states \( \rho_M \) and \( \rho_M^\perp \) by a projective two-component measurement \( \Pi_M = \{\Pi_{M0}, \Pi_{M1}\} \) on the qubit M. Using maximum likelihood estimation strategy, we can guess for each detection event the most likely result of the measurement \( \Pi_S \).

Our knowledge can be quantified as the fractional excess of the right guesses over wrong guesses in many such experiments repeated under identical conditions [5]. Using our expansion of \( \rho_{SM} \), the total knowledge is

\[
K(\Pi_M \rightarrow \Pi_S) = \sum_j |\text{Tr}_{\Pi_M}(w|\rho_M - w^\perp\rho_M^\perp|)\rangle|,
\]

where without the measurement \( \Pi_M \), the knowledge is \( K(\Pi_S) = w - w^\perp \). The largest value of knowledge over all \( \Pi_M \) was introduced as distinguishability

\[
D(\Pi_S) = \text{Tr}_{\Pi_M}(w|\rho_M - w^\perp\rho_M^\perp|).
\]

Further, we define a knowledge excess

\[
\Delta K(\Pi_M \rightarrow \Pi_S) = K(\Pi_M \rightarrow \Pi_S) - K(\Pi_S),
\]

where \( 0 \leq \Delta K(\Pi_M \rightarrow \Pi_S) \leq 1 \). It quantifies only that amount of the knowledge which exceeds the \textit{a priori} knowledge \( K(\Pi_S) \). The largest \( \Delta K(\Pi_M \rightarrow \Pi_S) \) over all \( \Pi_M \) can be considered as a distinguishability excess \( \Delta D(\Pi_S) \).

Intuitively, for a given mixed state \( \rho_{SM} \) of a two-qubit system the knowledge excesses are somehow restricted by the properties of the state. To derive a quantitative constraint, we will use the following expansion of the state:

\[
\rho_{SM} = \frac{1}{2}(1 \otimes 1 + \sum_i m_i |\sigma_i\rangle\langle \sigma_i| + \sum_j n_j |\sigma_j\rangle\langle \sigma_j| + \sum_{k,l=1}^2 t_{kl}|\sigma_k\rangle\langle \sigma_l|),
\]

where \( 1 \) stands for the identity operator; \( m_i, n_j \) are vectors in \( R^2 \); \( \sigma_i, \sigma_j = 1, 2, 3 \) are the standard Pauli operators. The coefficients \( t_{kl} \) form a real correlation matrix \( T \) and the vectors \( m_i \) and \( n_j \) determine the local states \( \rho_S = \frac{1}{2}(1 + \sum m_i |\sigma_i\rangle\langle \sigma_i|), \rho_M = \frac{1}{2}(1 + \sum n_j |\sigma_j\rangle\langle \sigma_j|). \)
is a subset of the states \( \tilde{\rho}_{SM} \) having a diagonal correlation tensor \( T = \text{diag}(r_{13}, r_{11}, r_{22}) \), with the following property \( r_{33} \geq r_{11}, r_{22} \). The local states are determined by the corresponding vectors \( \vec{m}_1 \) and \( \vec{n}_1 \). Any mixed state \( \rho_{SM} \) can be uniquely converted to a state \( \tilde{\rho}_{SM} \) using appropriate local unitary operations [6]. Thus, just two orderings of the diagonal elements, \( r_{11} \geq r_{22}, r_{11} \leq r_{22} \), remain to be discussed.

Let us suppose the measurements \( \tilde{\Pi}_S \) and \( \tilde{\Pi}_M \) are constructed from projectors to the vectors of the local bases in which \( |t_{i3}| \) is maximal. Then \( \Delta D(\tilde{\Pi}_3) = \Delta K(\tilde{\Pi}_M \rightarrow \tilde{\Pi}_S) = \max(0, |r_{i3} - |n_i|) \); for this choice of measurements \( \Delta K \) gets its maximal value over all \( \Pi_M \). Let us suppose the other pair of measurements. If \( r_{11} \geq r_{22}, r_{11} \leq r_{22} \), then \( \Delta D(\tilde{\Pi}_3) = \Delta K(\tilde{\Pi}_M \rightarrow \tilde{\Pi}_S) = \max(0, |r_{i1} - |n_i|) \). Simultaneously, we express the violation of any Bell’s inequalities employing the criterion from Ref. [4]: A state \( \tilde{\rho}_{SM} \) violates Bell’s inequalities if its maximal Bell factor \( B_{\text{max}} = 2 \sqrt{r_{11}^2 + r_{13}^2} \) lies in the interval \((2, 2\sqrt{2})\). Analogous results can be derived if \( r_{21} \leq r_{22} \). The maximal Bell factor is then \( B_{\text{max}} = 2 \sqrt{r_{22}^2 + r_{23}^2} \). Finally we obtain an inequality \( \Delta D(\tilde{\Pi}_3) + \Delta D(\tilde{\Pi}_3') \leq (B_{\text{max}}/2)^2 \) valid for an arbitrary state \( \tilde{\rho}_{SM} \). The equality occurs for states with zero a priori knowledge. For such states a nonzero knowledge can be obtained only though the measurement on \( M \).

Now we generalize these results to any state \( \tilde{\rho}_{SM} \) as well as for arbitrary measurements \( \Pi_S, \Pi_3, \Pi_M, \Pi_3' \), where \( \Pi_S, \Pi_3 \) are complementary measurements. As pointed out, any mixed two-qubit state can be uniquely prepared from some state \( \tilde{\rho}_{SM} \) (of a special form discussed above) by appropriate local unitary transformations \( U_S, U_M \) acting on qubits \( S \) and \( M \), respectively. Further, the transformation of the above chosen measurements \( \tilde{\Pi}_S \) and \( \tilde{\Pi}_M \) to arbitrary (but still complementary) measurements \( \Pi_S \) and \( \Pi_M \) corresponds effectively to the extra local unitary transformation \( U_{\Pi} \) acting on the qubit \( S \). Since distinguishabilities \( \Delta D(\Pi_3) \) and \( \Delta D(\Pi_3') \) are invariant under any local unitary transformation on the qubit \( M \), it is sufficient to take into account only a joint unitary transformation \( \tilde{U}_S = U_{\Pi} U_S \) acting on qubit \( S \). For any unitary transformations \( U \) there is a unique rotation \( O \) such that \( U(\vec{n} \cdot \vec{\sigma}) U^\dagger = (O \vec{n}) \cdot \vec{\sigma} \). If a state \( \tilde{\rho}_{SM} \) with diagonal \( \tilde{T} \) is subjected to the \( U_S \otimes U_M \) transformation its correlation matrix transforms as follows \( T = O_S \tilde{T} O_M^\dagger \) [6]. Thus a joint unitary transformation \( \tilde{U}_S \) can be represented as a transformation of the correlation tensor \( T = O_S \tilde{T} \), where \( O_S \) is a matrix of rotation in \( R^3 \) space.

First, we will explicitly calculate \( \Delta D(\Pi_3) \) and \( \Delta D(\Pi_3') \) for any mixed state using the transformation \( T = O_S \tilde{T} \). Assuming \( r_{11} \geq r_{22} \) we obtain \( \Delta D(\Pi_3) = \max(0, \sqrt{r_{13}^2 + r_{21}^2 + r_{11}^2 - |n_1|^2}) \) and \( \Delta D(\Pi_3') = \max(0, \sqrt{r_{13}^2 + r_{21}^2 + r_{11}^2 - |n_1|^2}) \). Then we straightforwardly get \( \Delta D^2(\Pi_3) + \Delta D^2(\Pi_3') \leq (B_{\text{max}}/2)^2 \). By analogous calculations we obtain the same result for \( r_{11} \leq r_{22} \). Finally, since \( \Delta K(\Pi_M \rightarrow \Pi_S) \leq \Delta D(\Pi_3) \) and \( \Delta K(\Pi_M' \rightarrow \Pi_S') \leq \Delta D(\Pi_3') \) we can conclude that

\[
\Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi_M' \rightarrow \Pi_S') \leq \left( \frac{B_{\text{max}}}{2} \right)^2. \tag{1}
\]

Thus the maximal Bell factor represents a nontrivial bound on the sum of the squares of knowledge excesses which can be extracted from a pair of measurements on the meter qubit. Assuming \( \Pi_M = \Pi_M' \) we can also derive an inequality analogous to that given in Ref. [5]: \( \Delta K^2(\Pi_M \rightarrow \Pi_S) + \Delta K^2(\Pi_M \rightarrow \Pi_S') \leq 1 \). Our analysis shows that for \( \Pi_M \neq \Pi_M' \) the unit value on the right-hand side may be overstepped. Note also that \( (B_{\text{max}}/2)^2 > 1 \) only if the state violates Bell inequalities. For details of the proofs see Ref. [7].

A natural question is how inequality (1) can be saturated. For the class of states with vanishing a priori knowledge for any measurements \( \Pi_S, \Pi_3 \) it can be saturated just by the appropriate choice of measurements \( \Pi_S, \Pi_3, \Pi_M, \Pi_3' \). In fact, it corresponds to the transformation of the given state to the state with diagonal correlation tensor. It was recently shown that there are such unique local filtering operations applicable on a single copy of a qubit pair that transform any two-qubit mixed state into a state which is (i) diagonal in Bell basis and (ii) has the Bell factor \( B_{\text{max}} \geq B_{\text{max}} \) [8]. Since these Bell-diagonal states have the both local states maximally disordered the a priori knowledge vanishes. Thus—because the inequality (1) is satisfied also after the filtering—we can always saturate it with the upper bound given by \( B_{\text{max}}^2 \) just by an appropriate choice of the measurements \( \Pi_S, \Pi_3, \Pi_M, \Pi_3' \) after the appropriate local filtering.

We have verified inequality (1) experimentally for two Werner states of qubits, \( |\Psi^-\rangle\langle\Psi^-| + [(1 - p)/4]| \) each qubit was represented by a polarization of a photon) [9]. The parameter of the first Werner state (\( p_1 = 0.82 \)) has been chosen so that the state was entangled and violated Bell inequalities, the parameter of the second one (\( p_2 = 0.45 \)) so that it was entangled but did not violate Bell inequalities. The scheme of our experimental setup is shown in Fig. 1. A krypton-ion cw laser (413.1 nm, 90 mW) is used to pump a 10-mm-long LiIO3 nonlinear crystal cut for degenerate type-1 spontaneous parametric down-conversion. The generated photons have horizontal linear polarizations. Different linear-polarization states are prepared by means of half-wave plates (A/2). The two photons impinge on two input ports of a beam splitter (BS) forming a Hong-Ou-Mandel (HOM) interferometer.
detector \[\text{PBS}\). It enables measurement in any linear-polarization basis parametrized by an angle \(\theta:\ \Pi_M = \{\Pi_M^+, \Pi_M^-\} = \{|\psi\rangle\langle\psi|, |\psi_1\rangle\langle\psi_1|\}, \text{ where } |\psi\rangle = \cos \theta |H\rangle + \sin \theta |V\rangle\]
and \( |\psi_1\rangle = \sin \theta |H\rangle - \cos \theta |V\rangle\). The angle \(\theta\) was set by a properly rotated half-wave plate. Similarly, two measurements on the signal qubit, \(\Pi_S\) and \(\Pi_S'\), were represented by polarization measurements in two bases rotated by \(45^\circ\): \(\Pi_S = \{\Pi_S^+, \Pi_S^-\} = \{|H\rangle\langle H|, |V\rangle\langle V|\times |V\rangle\langle H|, |H\rangle\langle V|\times |H\rangle\langle V|\} = \{|X\rangle\langle X|, |Y\rangle\langle Y|, |X\rangle\langle Y|, |Y\rangle\langle X|\}, \text{ where } |X\rangle = (|H\rangle + |V\rangle)/\sqrt{2} \text{ and } |Y\rangle = (|H\rangle - |V\rangle)/\sqrt{2}\). In practice, we measured coincidence rates between outputs \(\Pi_M^+\) and \(\Pi_S^-\) (it is denoted \(C^{++}\)), between \(\Pi_M^-\) and \(\Pi_S^+\) (it is denoted \(C^{-+}\)), etc. (the first sign concerns the \(M\) qubit, the second one the \(S\) qubit). Then the knowledge \(K(\theta) = |\text{Tr}_{MS}(\Pi_M^+| \Pi_S^- - \Pi_S^-)\rangle + |\text{Tr}_{MS}(\Pi_M^-| \Pi_S^+ - \Pi_S^+)\rangle|\) can be calculated from measured rates as follows: \(|C^{++} - C^{-+}| + |C^{-+} - C^{++}|/N\), \(N = C^{++} + C^{-+} + C^{+-} + C^{-+}\). Analogously, the \(a\ priori\) knowledge \(P = |\text{Tr}_{MS}(\Pi_S^- - \Pi_S^-)\rangle|\) can be obtained as \(|C^{++} + C^{-+} - (C^{++} + C^{-+})|/N\). The knowledge excess is given as \(\Delta K(\theta) = K(\theta) - P\). The quantities \(K'(\theta), P', \text{ and } \Delta K'(\theta)\) are obtained in the same way just with \(\Pi_S^+\) instead of \(\Pi_S\). The maximal violation of Bell inequalities, \(B_{\text{max}}\), can be obtained by measuring correlation functions for two different polarization bases on each side. Namely, for the Werner states one can choose those bases rotated by \(22.5^\circ\) and \(67.5^\circ\) (with respect to the vertical axis) on the one side and \(45^\circ\) and \(0^\circ\) on the other side: \(B_{\text{max}} = |C(22.5^\circ, 45^\circ) + C(67.5^\circ, 45^\circ) + C(22.5^\circ, 0^\circ) - C(67.5^\circ, 0^\circ)|\), \(C(\theta_1, \theta_2)\) is estimated from the measured data as \((C^{++} + C^{-+} - C^{++} - C^{-+})/N\). Let us note that for Werner states the theoretical predictions of regarded quantities read \(K = |p| \cos(2\theta)|, K' = |p| \sin(2\theta)|, P = P' = 0, B_{\text{max}} = p^2/2\). Clearly, maximal value of \(\Delta K^2(\theta) + \Delta K'^2(\theta')\) should appear for \(\theta = 0^\circ\) (and \(90^\circ\), \(\theta' = 45^\circ\).

![FIG. 1. Experimental setup.](image)

![FIG. 2. The squares of the knowledge excesses and their sum measured for the Werner state with \( p = 0.82\). Symbols show experimental values, full lines theoretical predictions (for \( p = 0.82\)).](image)
The following graphs display our experimental results. In Fig. 2 there are plotted the squares of the knowledge excesses $\Delta K^2 / 0.0133 \# / 0.0134$ and their sum measured for the Werner state with parameter $p = 0.0025$ (this parameter was estimated from the best fit accordingly to the theoretical predictions for Werner states). The error bars show statistical errors. The accuracy of polarization-angle settings was about $\pm 0.0006$. Figure 3 shows the sum $\Delta K^2 / 0.0133 \# / 0.0134$ as a function of two angle variables for the same Werner state. The maximal displayed value of the vertical axis shows the measured value of $(B_{\text{max}}/2)^2$.

The maximal measured Bell factor is $B_{\text{max}} = 2.36 \pm 0.02$ what is in a good agreement with the theoretical value for $p = 0.818 \pm 0.007$ that equals $2.314 \pm 0.020$. The same kind of measurement is presented in Fig. 4 but now for the Werner state with $p = 0.45$. The corresponding measured maximal Bell factor is $B_{\text{max}} = 1.32 \pm 0.02$ (theoretical value for $p = 0.453 \pm 0.008$ is $1.281 \pm 0.023$). As can be seen, for both measured states the experiment has verified inequality (1).

The measurement on the one of two correlated particles gives us the power of prediction of the measurement results on the other one. Of course, one can never predict exactly the results of two complementary measurements at once. However, knowing what kind of measurement we want to predict on signal particle, we can choose the optimal measurement on the meter particle. But there is still a fundamental limitation given by the sort and amount of correlations between the particles. Both of these kinds of constraints are quantitatively expressed by our inequality (1). The limitation stemming from mutual correlation of particles manifests itself by the maximal Bell factor appearing in the inequality. We have proved this inequality theoretically as well as tested it experimentally.

We appreciate discussions with J. Fiurášek and L. Míšta, Jr. This research was supported by Projects No. 202/03/D239 of GACR and No. LN00A015 and No. CEZ:J14/98 of the Ministry of Education of the Czech Republic.