JOURNAL OF MODERN OPTICS, 2004, VOL. ??, NO. ??, 1–4

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# Uncertainty relations from Fisher information

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(*Received 22 October 2003*)

**Abstract.** Heisenberg uncertainty relations yield sometimes rather trivial statements. For instance, the uncertainty of the transversal momentum of a particle behind a rectangular slit is infinite, which yields $\infty \cdot \Delta x \geq \hbar/2$. Motivated by the estimation theory we will show that a ''good'' uncertainty relation can be recovered provided one gives the uncertainties a slightly different meaning. This elementary example will also hint at the importance of the information theory for the quantum physics.

## 1. Diffraction on the slit

Let us consider the standard setup and standard argumentation used in the elementary textbooks of quantum mechanics for the exposition of Heisenberg uncertainty relations [1]. For simplicity assume 1D geometry of a single slit sketched in figure 1.

The particle goes through the slit impinging on the position sensitive screen behind the slit. According to the de Brogli hypothesis it will effectively behave as a wave with the de Brogli wave length $\lambda = h/p$ where $p$ is the impulse of the particle. Using a simple geometrical argumentation, each detection event on the screen may be used for inferring the direction of the incoming particle. Hence this scheme could be considered as a measuring device for determining the impulse. Invoking the effect of diffraction, the quantum nature of particles will be manifested by a diffraction pattern registered on the screen. Assuming the illumination by a plane wave, the state describing the particle behind the slit reads

$$\psi(x) = \begin{cases} \exp(ik_x x)/\sqrt{a}, & |x| \leq a/2, \\ 0, & |x| > a/2. \end{cases} \tag{1}$$

Here $k_x = k \sin\theta$ is the component of the wave vector $k = 2\pi/\lambda$ orthogonal to the optical axis, and $\lambda$ is the wavelength of the particle. Denoting the detected position of the particle on the screen by $\xi$, the probability of the detection of the particle at point $\mu$ in the far reach zone is given by the square of its Fourier transformed wave function,

$$p(\mu|\nu) = \frac{1}{\pi} \operatorname{sinc}^2(\mu - \nu). \tag{2}$$

Here the dimensionless quantities used are $\mu = (ak/2d)\xi$ and $\nu = a/2k_x$. Taking the first minimum of function (2) for defining its ''spatial extent'', the half-width of the probability distribution is then determined as $\Delta k_x = 2\pi/a$. Considering

*Journal of Modern Optics* ISSN 0950–0340 print/ISSN 1362–3044 online 
http://www.tandf.co.uk/journals
DOI: 10.1080/09500340410001663954

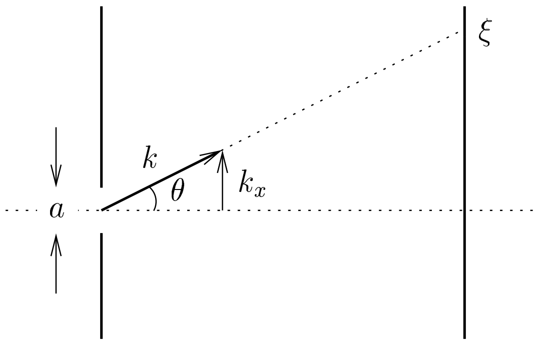


Figure 1. Geometry of the diffraction on the slit.

further that in the plane of the slit the location of the particle is known to be within the half-width of $\Delta x = a/2$, the expected uncertainty relation reads

$$\Delta p_x \Delta x \approx h/2, \tag{3}$$

where $p = \hbar k$. This is often considered as a painless, quick and intuitive way of formulating the uncertainty principle. Although relation (3) resembles the famous Heisenberg uncertainty relation of quantum theory,

$$\Delta p_x \Delta x \geq \hbar/2, \tag{4}$$

one should realize that the uncertainties in (4) are strictly defined as the root-mean-square variances of observables unlike those in equation (3). This becomes crucial in the case of the probability distribution (2) whose variance is infinite due to its heavy tails [2]. Obviously, as the uncertainty of the momentum is infinitely large, inequality (4) is trivially satisfied.

There are several ways for circumventing this obstacle, for example by an alternative definition of the proper resolution measure, or by invoking entropic uncertainty relations [3]. Motivated by recent progress in the understanding of the key role played by information in quantum theory [? ?], we will proceed in 1 a different way and use the concept of the Fisher information. This information imposes ultimate limitations on measurements, and as such it can be used to describe the uncertainty relations in a generalized sense.

## 2. Fisher information of interference patterns

The build-up of an interference pattern is governed by a probabilistic law, where the intensity (2) plays the role of a probability distribution governed by a true parameter $\nu$. After registering $N$ particles, this parameter can be estimated. Of course, different estimators will, in general, have different errors. However, an important theorem of probability theory states that the error of any unbiased estimator of $\nu$ is lower bounded by the Fisher information $I$ as follows,

$$(\Delta\nu)^2 \geq 1/I. \tag{5}$$

Assuming that the statistics of the experiment is multinomial (the discretization comes from the binning of the position measurement), the Fisher information reads,

$$I = NF = N \sum_{\mu} \frac{p'^2(\mu|\nu)}{p(\mu|\nu)}, \tag{6}$$

where the prime denotes derivation with respect to $\nu$. Relation (5) is the well-known Cramer-Rao lower bound (CRLB) [6, 7].

To achieve the resolution given by equation (5) it is necessary to register a large number of particles $N \gg 1$. Since the information provided by the optimal estimator grows linearly with increasing number of particles, it is possible to define the information gained per particle as $F = I/N$. This quantity provides the ultimate resolution corresponding to a single ''average'' particle from the bunch of
**2** registered events [?].

Let us apply this theory to the interference pattern behind the slit. It is easy to calculate all the respective quantities

$$(\Delta\mu)^2 = \left(\frac{a}{2\hbar}\right)^2 (\Delta p_x)^2, \quad (\Delta x)^2 = \frac{a^2}{12}, \quad F = \frac{4}{\pi}\int \mathrm{d}\mu \left(\frac{\mathrm{d}}{\mathrm{d}\mu}\operatorname{sinc}\mu\right)^2 = \frac{4}{3}. \tag{7}$$

Interestingly, the CRLB (5) reproduces exactly the expected Heisenberg uncertainty relations for the impulse and position of the particle going through the slit. Indeed, substituting equations (7) into CRLB (5) with $I$ replaced by $F$ we find that

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}. \tag{8}$$

There is a simple relationship between the Fisher information and the variances of complementary variables. Considering the momentum representation, $\psi(p) = \langle p|\psi\rangle$, the Fisher information may be rewritten in the form

$$\begin{aligned} F &= \int \mathrm{d}p \frac{[\psi(p)^{*\prime}\psi(p) + \psi(p)'\psi(p)^*]^2}{\psi(p)\psi^*(p)} = \int \mathrm{d}p \frac{\langle p|\,[(\Delta\hat{X})/(i\hbar), |\psi\rangle\langle\psi|]\,|p\rangle^2}{\psi(p)\psi^*(p)} \\ &= \frac{4(\Delta x)^2}{\hbar^2} - \int \mathrm{d}p\,\psi(p)\psi^*(p)\left[\frac{\partial}{\partial p}\arg\psi + \frac{1}{\hbar}\langle x\rangle\right]^2. \end{aligned} \tag{9}$$

This is the Fisher information associated with one average particle prepared in state $\psi(p)$. The corresponding momentum uncertainty *per particle* predicted by the Cramér-Rao lower bound is

$$(\Delta p)^2 \geq 1/F. \tag{10}$$

Equations (9) and (10) imply a Heisenberg-like uncertainty relation with the difference that $p$ in equation (10) is *an estimator* of the particle momentum. This inequality will take exactly the form of a Heisenberg uncertainty relation whenever the phase of the wave function in $p$-representation exhibits a linear dependence on the momentum, which gives a wide class of minimum uncertainty states for the information uncertainty relation on the slit.

Fisher information could also provide a new insight into the problem of quantum complementarity. Given a state $\rho$ we can ask which unitary operation $\exp(-i\xi X)\rho\exp(i\xi X)$ applied to this state can be most accurately revealed by measuring an observable $A$. Obviously, the most favorable situation arises when the operators $X$ and $A$ satisfy the relation

$$\exp(i\xi X)A\exp(-i\xi X) = A + \xi, \tag{11}$$

which means that they obey the canonical commutation relations $[A, X] = i$. Unfortunately, such direct measurements do not always exist, as for example in the finite dimensional Hilbert spaces. But even in those cases, an operational formulation can be given in the framework of the estimation theory where the question can be rephrased as follows: what generator maximizes the Fisher information corresponding to angle $\xi$, provided that operator $A$ has been measured?

The answer to this question will be given elsewhere. Let us just mention that for qubits, such an operational definition of complementarity based on the estimation theory coincides with the usual definition.

We have formulated several arguments in favor of Fisher information and its applications to quantum problems. Fisher information provides the ultimate limitation for quantum measurements and as such provides also a nontrivial link between the theory of statistics and quantum theory. Since quantum theory is more ''operational'' than perhaps any other physical theory, this may yield new interesting insights into its fundamentals.

## Acknowledgments

This work has been supported by projects LN00A015 and J14/98 of the Czech Ministry of Education, by the EU grant QIPC, Project No. IST-1999-13071 (QUICOV), and by Czech-Italian project No. 29, ''Decoherence and Quantum Measurement''.

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