QUANTUM TOMOGRAPHY: RENORMALIZATION OF INCOMPATIBLE OBSERVATIONS

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Abstract

Standard deterministic techniques for quantum state reconstruction, as for example optical homodyne tomography, photon chopping, unbalanced homodyning etc., are based on the deterministic inversion of measured data. Since the frequencies obtained in realistic experiments always differ from probabilities predicted by quantum theory due to fluctuations, imperfections and realistic restrictions, the algorithm of inversion cannot guarantee the positive definiteness of the reconstructed density matrix. Hence the estimation of the noises may appear as doubtful.

Quantum states may be successfully reconstructed within quantum and information theories using the maximum likelihood estimation. The question of deterministic schemes: " What quantum state is determined by that measurement?" is replaced by the formulation consistent with quantum theory: " What quantum state(s) seems to be most likely for that measurement ?" Nonlinear equation for reconstructed state is formulated. An exact solution may be approached by subsequent iterations. Reconstruction is formulated as a problem of proper normalization of incompatible (nonorthogonal) measurements. The results obtained by this novel method may differ significantly from the standard predictions. Data are fitted better keeping the constraint of positive definiteness of reconstructed density matrix. However, this interpretation may enlarge uncertainty in prediction of quantum state in comparison with deterministic schemes, since, in general, there is a whole family of states which fit the measured data equally well. The novel technique is nonlinear and the reinterpretation of existing reconstruction schemes represents an advanced program.

INTRODUCTION

Quantum theory brings information about observable events on the most fundamental level currently available. The statistical nature of almost all quantum phenomena seems to be its characteristic feature. This intrinsic uncertainty cannot be considered in accordance with the "classical" experience as a lack of knowledge about the internal structure of the system, since this does not exhaust the richness of the quantum world. The intrinsic uncertainty is hidden in the wave function, the origin of which remains unknown and unexplained by quantum theory. The pragmatic interpretation of quantum theory concentrates on the observable aspects, which may be successfully addressed within existing techniques. Determination of quantum state on the basis of the performed measurement may be acknowledged as one of those topical problems.

Although the history of state reconstruction may be traced back to the early days of quantum mechanics to Pauli problem [1], only quantum optics opened the new era. Theoretical prediction of Vogel and Risken [2] was closely followed by the experimental realization of the suggested algorithm by Smithey et. al. [3]. Since that time many improvements and new techniques have been proposed [4,5] and similar techniques are currently being used also in atomic physics as quantum endoscopy [6]. The potential gain of this treatment is tremendous. Provided that certain quantum measurement enables us to determine the wave function of the system, the statistics of any further possible measurement may be forecast. The quantum state reconstruction plays therefore the role of a universal measurement. and is considered nowadays as a standard technique .

Nevertheless, there are some potential problems associated with standard treatment. Available realistic measurements are always limited as far as the amount and accuracy of data is concerned. Consequently, any scheme for posterior estimation is affected by these imperfections. Particularly, the standard techniques based on inversion of quantum prediction do not preserve the semipositive definiteness of reconstructed density matrix, a necessary condition of quantum state definition. This may be accomplished using the statistical approach. There are several proposals based on various statistical concepts, including Bayesian approach [8], Maximum Entropy [9] principle or the Maximum Likelihood estimation (MaxLik). They all release the relation between the observed data and the quantum state generating them. The MaxLik estimation will be focused in this contribution. Its formulation follows closely the motivation of standard treatments.

The technique of MaxLik estimation is widely used in various branches of technology and science [10]. It is usually used as a tool for fitting a few parameters which maximizes the likelihood under the given constraints. In quantum theory, MaxLik has already been used for quantum phase estimation [11,12] and recently, it has been adopted for estimation of more parameters as well [13,14]. Nevertheless, in all these applications additional constraints were crucial for successful application of this method. For example, this is done by restrictions put on the dimension of the problem [13], or by additional assumptions about the nature of the noise [14]. Without these conditions the problem of MaxLik estimation is considered as intractable due to the multidimensional nonlinear optimization. Nevertheless, quantum theory of MaxLik estimation without any other additional assumptions is addressed in this contribution. As a result, this complex optimization can be interpreted as quantum prediction for renormalized projectors. Hence, the procedure of mathematical statistics may be interpreted purely in the language of quantum theory. This feature indicates another close and fundamental relation between geometry of Hilbert space and concepts of mathematical statistics [15,16]. Various issues of this novel treatment are explained in this contribution in the form of a dozen frequently asked questions concerning quantum tomography.

A DOZEN FREQUENTLY ASKED QUESTIONS

1. WHAT IS THE QUANTUM TOMOGRAPHY ?
   Tomography is routinely used in medicine. A picture of body is obtained by a synthesis of many various X-ray projections of the object. The registration of rotated quadrature operator $\hat{x}_\theta = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/2$ using homodyne detection represents an analogy to the absorption of X–rays in medicine. Complete probability $P(x_\theta, \theta)$ determines, for example, the Wigner function of quantum state by the Radon transformation [2]
   $$W(\alpha_r, \alpha_i) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx|\eta|d\eta \int_0^\pi d\theta P(x,\theta) e^{i\eta x - i\alpha_r \cos\theta - i\alpha_i \sin\theta}.$$
   Similarly, there are many other experimental techniques in quantum theory. That enables us to observe "various faces" of a quantum system: homodyne detection [3,17,18], photon chopping [19], quantum state measurement via heterodyne detection [20], unbalanced homodyning [21], direct probing by photon counting [22], quantum endoscopy [6], etc.

2. WHAT IS THE PRAGMATIC INTERPRETATION OF QUANTUM TOMOGRAPHY?
   Quantum tomography is nothing else than a theory of conditional measurements. Already performed experiments condition the future outcomes of different forthcoming measurements done on the same system.

3. WHAT IS THE COMMON MATHEMATICAL FORMULATION OF STANDARD APPROACHES?
   Standard approaches are based on the formal inversion of quantum prediction
   $$\langle y_i|\hat{\rho}|y_i\rangle = f_i,$$
   where $|y_i\rangle$ represent the registered projectors and f_i are their counted frequencies. Particularly, in the case of tomography based on homodyne detection, the probability $P(x,\theta)$ is simply replaced by discretized frequencies $f(x_i, \theta_j)$.

4. ARE THERE ANY APPARENT FLAWS IN STANDARD APPROACHES?
   Yes. The reconstructed object is not a density matrix $\hat{\rho} \geq 0$ in general! Hence, some probabilities are predicted as negative. Standard approaches do not describe the statistical nature of quantum observations properly and do not distinguish sufficiently what can and what cannot be predicted from the given data [7].

5. WHAT IS THE REMEDY AGAINST THE DIFFICULTIES OF STANDARD APPROACHES?
   An approach based on statistical interpretation. The detected data could be generated by nearly any quantum state, nevertheless, with various likelihoods. Likelihood is given by the product of probabilities for all independently counted outcomes (enumerated by index i here)
   $$\mathcal{L} = \prod_i p_i^{n_i}$$
   provided that data enumerated by index i are registered n_i times, $\sum n_i = n$.

6. COULD THE NOVEL APPROACH TO QUANTUM TOMOGRAPHY BE DEMONSTRATED ON A SIMPLE STATISTICAL MODEL?
   Yes, on the model of estimation of prior probabilities. Assume that data n_i are generated with some prior probabilities $p_i, \sum_i p_i = 1$, which should be found from the data. The likelihood for associating the data $\{n_i\}$ with some probabilities $\{p_i\}$ is given by the multinomial distribution

$$\mathcal{P}(\{p_i\}|\{n_i\}) = n! \prod_i \frac{1}{n_i!} p_i^{n_i}.$$

   The most probable guess is given by frequencies $f_i = n_i/n$.

7. IS THE STATISTICAL APPROACH SOMEHOW RELATED TO ENTROPY?
   Yes, to the relative entropy (Kullback–Leibler divergence). The logarithm of the ratio of likelihood functions
   $\mathcal{P}(\{f_i\})/\mathcal{P}(\{p_i\})$ yields the Kullback–Leibler divergence

$$K(f_i/p_i) = n \sum_i f_i \log \frac{f_i}{p_i}.$$

   This equals up to the sign to the relative entropy [23].

8. WHAT IS THE MAXIMUM LIKELIHOOD ESTIMATION?

   MaxLik estimation searches for the state which provides the largest likelihood for the given data. Equivalently, such states minimize the relative entropy for the given data. In the above mentioned example the MaxLik estimation of probabilities is given by frequencies f_i. However, there are some significant differences in quantum theory:

   - The estimated probabilities depend on the same quantum state and are therefore not independent.
   - The probabilities are not normalized to one since projectors need not be complete.

9. WHAT IS THE MATHEMATICAL FORMULATION OF MAXLIK ESTIMATION?

   Probabilities are given in quantum theory as

$$p_i = \langle y_i|\hat{\rho}|y_i\rangle.$$

   Maximization of likelihood on the class of possible states (density matrices) corresponds to the nonlinear operator equation

$$\hat{R}(\hat{\rho})\hat{\rho} = \hat{\rho}.$$

   Unknown density matrix is assumed in its diagonal form

$$\hat{\rho} = \sum_k r_k |\phi_k\rangle\langle\phi_k|,$$

   and operator $\hat{R}$ is given as

$$\hat{R} = \sum_i \frac{f_i}{\rho_{ii}} |y_i\rangle\langle y_i|,$$

$$\rho_{ii} = \sum_k r_k |\langle \phi_k | y_i \rangle|^2.$$

Reconstruction is done in the subspace where operator $\hat{R}$ equals to the identity operator [24] .

10. WHAT IS THE PHYSICAL INTERPRETATION OF MAXLIK RECONSTRUCTION?
MaxLik estimation may be interpreted as renormalization of incompatible (noncommuting) observables for which the synthesis of various projections is done. Assume the rescaling of the projectors as
$$|y_i\rangle\langle y_i| \rightarrow |y_i'\rangle\langle y_i'| = \frac{f_i}{\rho_{ii}} |y_i\rangle\langle y_i|.$$
The rescaled projectors satisfy the quantum prediction
$$\langle y_i'|\hat{\rho}|y_i'\rangle = f_i.$$
The operator $\sum_i |y_i'\rangle\langle y_i'|$ characterizes the overlapping of rays $|y_i'\rangle$. Hence, the MaxLik reconstruction reproduces the quantum prediction for suitable renormalized projectors. This is achieved on a subspace–field of view–enclosed by condition $\hat{R} = \hat{1}$.

11. WHAT ARE THE PHYSICAL CONSEQUENCES OF MAXLIK ESTIMATION?
MaxLik approach generalizes the standard treatment. Whenever the latter one has a solution on the manifold of density matrices, then this is the solution of the former one as well. Besides this, MaxLik estimation provides a whole family of extremum states not distinguishable by the given measurement. Averaging over this family enhances the uncertainty of state prediction. MaxLik reconstruction of diagonal elements of density matrix has been already done for homodyne detection with fluctuating phases [25,26].

12. DOES THIS FIELD OF VIEW HAVE ANY SIMPLE INTERPRETATION?

Realistic data can never provide complete information about quantum state in infinite dimensional Hilbert space. Any prediction should be restricted to certain field of view. Assume a simple example of reconstruction of diagonal elements of density matrix via photon counting with ideal detector. Suppose n times repeated counting, always with zero registered photoelectrons. The "standard" prediction of quantum state reads $\hat{\rho} = |0\rangle\langle 0|$, where $|0\rangle$ denotes the vacuum state. Nevertheless this interpretation is not the only one. Assume an additional classical noise represented by a projector into the strong coherent state $\hat{\mathcal{N}}_\epsilon = |\frac{\alpha}{\sqrt{\epsilon}}\rangle_{coh}\langle\frac{\alpha}{\sqrt{\epsilon}}|$ appearing with the negligible probability ϵ. The state
$$\hat{\rho}_\epsilon = (1-\epsilon)\hat{\rho} + \epsilon \hat{\mathcal{N}}_\epsilon$$
cannot be distinguished from the standard one for sufficiently small $\epsilon < 1/n$. This may appear as crucial for some observations. For example, the average numbers of particles differ significantly for both the states. Field of view is specified as the subspace where operator $\hat{R}$ reproduces the identity operator. This is evident in the case of orthogonal measurements but rather nontrivial in the case of nonorthogonal measurements.

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REFERENCES

1. S. Weigert, *Phys. Rev. A* 45: 7688 (1992).
2. K. Vogel, H. Risken, *Phys. Rev. A* 40: 2847 (1989).
3. D. T. Smithey, M. Beck, M. G. Raymer, A. Faridani, *Phys. Rev. Lett.* 70: 1244 (1993).
4. U. Leonhardt, "Measuring of the Quantum State of Light," Cambridge Press, Cambridge, (1997).
5. D.– G. Welsch, W. Vogel, T. Opatrný, Homodyne detection and quantum state reconstruction, *to appear in* "Progress in Optics " , E. Wolf, ed., Kluwer Publishing, Amsterdam (1999).
6. S. Wallentowitz, W. Vogel, *Phys. Rev. Lett.* 75: 2932 (1996); P. J. Bardoff, C. Leichtle, G. Schrade, W. P. Schleich, *Phys. Rev. Lett.* 77: 2198 (1996); D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, D.J. Wineland, *Phys. Rev. Lett.* 77: 4281 (1996).
7. Z. Hradil, *Phys. Rev. A* 55: R1561 (1997).
8. K. R. W. Jones, *Phys. Rev. A* 50: 3682 (1994).
9. V. Bužek, G. Adam, G. Drobný, *Phys. Rev. A* 54: 804 (1996).
10. R. Barlow, "Statistics," J. Wiley, Chichester (1989).
11. J. H. Shapiro, S. R. Shepard, N. C. Wong, *Phys. Rev. Lett.* 62: 2377 (1989).
12. A. S. Lane, S. L. Braunstein, and C. M. Caves, *Phys. Rev. A* 47: 1667 (1993).
13. D. Mogilevtsev, Z. Hradil, J. Peřina, *Rev. Mod. Opt.* 44: 2261 (1997).
    D. Mogilevtsev, Z. Hradil, J. Peřina, *Quantum Semiclass. Opt.* 10: 345 (1998).
14. S. M. Tan, *Rev. Mod. Opt.* 44: 2233 (1997).
15. W. K. Wooters, *Phys. Rev. D* 23: 357 (1981).
16. S. L. Braunstein, C. M. Caves, *Phys. Rev. Lett.* 72: 3439 (1994).
17. G. M. D' Ariano, U. Leonhardt, H. Paul, *Phys. Rev. A* 52: R1801 (1995).
18. S. Schiller, G. Breitenbach, S. F. Pereira, T. Mü ller, J. Mlynek, *Phys. Rev. Lett.* 77: 2933 (1996).
19. H. Paul, P. Törma, T. Kiss, I. Jex, *Phys. Rev. Lett.* 76: 2464 (1996).
20. M. G. A. Paris, *Phys. Rev. A* 53: 2658 (1996).
21. S. Wallentowitz, W. Vogel, *Phys. Rev. A* 53:4528 (1996).
22. K. Banaszek, K. Wódkiewicz, *Phys. Rev. Lett.* 76: 4344 (1996).
23. C. M. Caves, P. D. Drummond, *Rev. Mod. Phys.* 66: 481 (1994).
24. Z. Hradil, J. Summhammer, H. Rauch, not published.
25. K. Banaszek, *Phys. Rev. A* 57: 5013 (1998); K. Banaszek, *acta phys. slov.* 48: 195 (1998).
26. Z. Hradil, R. Myška, *acta phys. slov.* 48: 199 (1998).