

Uncertainty relations on the slit and Fisher information

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Diffraction on the slit can be interpreted in accordance with the Heisenberg uncertainty principle. This elementary example hints at the importance of the information theory for the quantum physics. The role played by one particularly interesting measure of information — the Fisher information — in quantum measurements is further discussed in the context of quantum interferometry.

Quantum mechanics became a standard tool not only of physicists, but almost any scientist is familiar with at least some of its concepts. Traditionally it has provoked and attracted attention of broader community due to sometimes paradoxical implications on the fundamental and philosophical issues of the Nature. That is why teaching of quantum mechanics is a rewarding but not an easy task. There is perhaps no other field of physics encumbered by so many misconceptions and misinterpretations as quantum theory is. Every student realizes very quickly that depending on the degree of profundity there are no simple questions and answers here. Recently there was a live discussion concerning the teaching of quantum mechanics¹ followed by a comment² addressing a misunderstanding appearing in many elementary textbooks of quantum theory, namely the exposition of the Heisenberg uncertainty relation on the slit. We do not want to speculate whether it is better on the introductory level to suppress the complexity of the problem and present a simple but physically inadequate explanation, or to build a consistent but terse theory from the very beginning. This is certainly a deep problem, solution of which depends on the teacher. The purpose of this contribution is however different. The analysis of the diffraction of the wave (particle) on the slit will be used as a link between the quantum mechanics and mathematical statistics. We will provide a physical motivation for the concept of the Fisher information and will reconsider the problem of uncertainty relations related to the quantum interpretation of the interference pattern in wave theory. In our opinion, both these issues deserve attention: Fisher information comes from the mathematical statistics and as such it seems to do very little with quantum theory. That is perhaps why this topic is missing in standard textbooks. This may change in the future since now the crucial role of information physics is widely recognized, and there exist serious attempts to derive and explain the physical laws of nature and quantum theory itself from the theory of information^{3,4,5}.

The second goal of our approach is to formulate alternative inequalities for complementary observables motivated by the Fisher information and apply them to particles passing through the slit. This intuitively clear and illustrative example will demonstrate the incompatibility of the position and impulse observables in quantum theory. It is intriguing to note that this approach is also relevant for such up to date problems as quantum inter-

ferometry and quantum lithography^{6,7,8}.

I. DIFFRACTION ON THE SLIT

Let us consider the standard setup and standard argumentation used in the elementary textbooks of quantum mechanics for the exposition of Heisenberg uncertainty relations⁹. For simplicity assume 1D geometry of a single slit sketched in Fig. 1. The particle goes through the slit impinging on the position sensitive screen behind the slit. According to the de Brogli hypothesis it will effectively behave as a wave with the de Brogli wave length $\lambda = h/p$, p being the impulse of the particle. Using a simple geometrical argumentation, each detection event on the screen may be used for inferring the direction of the incoming particle. Hence this scheme could be considered as a measuring device for determining the impulse. Invoking the effect of diffraction, the quantum nature of particles will be manifested by a diffraction pattern registered on the screen. This effect will enhance the uncertainty of the inferred impulse. This qualitative reasoning can easily be accompanied by the corresponding wave picture that describes each detected single event. Assuming the illumination by a plane wave, the state describing the particle behind the slit reads

$$\psi(x) = \begin{cases} \exp(ik_x x)/\sqrt{a}, & |x| \leq a/2, \\ 0, & |x| > a/2. \end{cases} \quad (1)$$

Here $k_x = k \sin \theta$ is the component of the wave vector $k = 2\pi/\lambda$ orthogonal to the optical axes, and λ is the wavelength of the particle. Denoting the detected position of the particle on the screen by ξ , the probability of the detection of the particle in the far reach zone is given by the square of its Fourier transformed wave function,

$$p(\mu) = \frac{1}{\pi} \text{sinc}^2(\mu - \nu). \quad (2)$$

Here the dimensionless quantities used are $\mu = \frac{ak}{2a} \xi$ and $\nu = a/2k_x$. Naturally, such a detection visualizes the transversal momentum of the particle k_x impinging on the slit. The probability manifests one distinct peak at $\mu = \nu$. The standard interpretation relies on a geometrical argumentation. Taking the first minimum of the function (2) for defining its “spatial extent,” the half-width of the probability distribution is then determined

as $\Delta k_x = \frac{2\pi}{a}$. Considering further that in the plane of the slit the location of the particle is known to be within the half-width of $\Delta x = a/2$, the expected uncertainty relation reads

$$\Delta p_x \Delta x \approx \hbar/2, \quad (3)$$

where $p = \hbar k$. This is often considered as a painless, quick and intuitive way how to formulate the uncertainty relation, more so, because the relation (3) resembles the famous uncertainty relation of quantum theory ascribed to Heisenberg

$$\Delta p_x \Delta x \geq \hbar/2, \quad (4)$$

though its exact derivation in the present form has been done by Kennard¹⁰. One must however realize that contrary to the relation (3), the uncertainties in (4) are strictly defined as the root-mean-square variances of observables. Due to the formal similarity between both the relations one may be tempted to interpret the relation (3) as an approximation of the rigorous uncertainty principle (4). Unfortunately, this would not be correct, since, the whole derivation of (3) stands on the shaky ground. The error measures appearing in (3) have nothing to do with standard variances. This becomes crucial in the case of the probability distribution (2) whose variance is infinite due to its heavy tails. This problem is well known¹¹, and usually ascribed to the discontinuity of the wave function after the passage of the particle through the slit, and in fact, it disqualifies the example of the slit from all exact considerations. Obviously, as the uncertainty of the momentum is infinitely large, inequality (4) is trivially satisfied, in this case. There are several ways how to circumvent this obstacle, for example by alternative definitions of the proper resolution measure, or by invoking entropic uncertainty relations¹².

In the following exposition we will keep the example of the diffraction on a slit as a toy example and employ it for introducing the Fisher information. This information defines the ultimate limitations of measurements, and as such it can be used to describe the uncertainty relations in the generalized sense. As an interesting byproduct, the resolution of the current quantum interferometric techniques will be evaluated.

II. FISHER INFORMATION

The interference pattern registered behind the slit may be interpreted within a different framework. The build-up of an interference pattern is governed by a probabilistic law, where the intensity (2) plays the role of a probability distribution. Instead of a single-particle detection discussed in the previous section, let us consider the information accumulated in the detection of the full interferometric pattern created by altogether n particles. It does not matter whether they arrived in a single shot or one by one in the course of a subsequently repeated

experiment. The identity of detected particles is disregarded and this is the only additional assumption with respect to the previous case. Let us develop the statistical description capable of handling a generic statistical model, which will afterwards be applied to the problem of the diffraction on the slit.

Assume that generic data denoted by x are registered with the frequency of occurrence of n_x , and let us denote the total number of particles by n . For the concreteness, the variable x is considered to take discrete values. Moreover, we assume that the values x occur with the probability $p_x(\theta)$, where θ represents an unknown true value of a certain parameter. The purpose of the whole treatment is to infer the true value of this parameter as faithfully as possible from the registered data x . This is the general estimation scheme, which of course, might be adopted to the case of particles impinging on a screen as well. In particular, provided that the detection of each single particle is evaluated separately, $n = 1$, this scheme reduces to the above mentioned measurement of the impulse. Let us denote by θ a function of the registered data, which will be used for the estimation of the true value of the parameter of interest $\bar{\theta}$. This function is called estimator in the mathematical statistics and there are many nonequivalent ways how to construct it. Let us describe the maximum likelihood (ML) estimator adopted to the evaluation of the accumulated data set n_x . Since registrations of individual particles are independent events, the likelihood that the actual value of the parameter was θ conditioned upon the registered data is proportional to the product of individual probabilities,

$$\mathcal{P}(\theta|\{n_x\}) \propto \exp\left\{\sum_x n_x \ln p_x(\theta)\right\}. \quad (5)$$

ML estimator is given by such a value of θ which maximizes this function. Let us estimate its uncertainty. Provided that the total number n of registered particles is large, the registered data can be replaced by the expected number of detected particles, $n_x = n p_x(\bar{\theta})$. The likelihood (5) can then be expanded in a power series in the neighborhood of this true value,

$$\begin{aligned} \mathcal{P} &\propto \exp\left\{n \sum_x p_x(\bar{\theta}) \ln p_x(\theta)\right\} \\ &\approx \exp\left\{n \sum_x p_x(\bar{\theta}) \ln p_x(\bar{\theta}) \right. \\ &\quad \left. - \frac{n}{2} \sum_x \frac{p_x'^2(\bar{\theta})}{p_x(\bar{\theta})} (\theta - \bar{\theta})^2 + \dots\right\}. \end{aligned} \quad (6)$$

Its meaning is obvious. For sufficiently large number of particles the ML estimator fluctuates around the true value of the parameter within the error $1/F$,

$$F_\mu = \sum_x \frac{p_x'^2(\bar{\theta})}{p_x(\bar{\theta})}. \quad (7)$$

F is the Fisher information, which characterizes the root mean square error of the inferred value of the parameter

from its true value. Significantly, the Fisher information represents the ultimate limit for the resolution of any unbiased estimator. This relation is known as the Cramer-Rao lower bound^{13,14}. For its importance we repeat its derivation here¹⁵. Noticing that the mean value of any unbiased estimator equals the true value,

$$\sum_x \theta p_x(\bar{\theta}) = \bar{\theta}, \quad (8)$$

and using the normalization condition

$$\sum_x p_x(\bar{\theta}) = 1, \quad (9)$$

this inequality can be derived by differentiating Eqs. (8) and (9) with respect to $\bar{\theta}$, multiplying the latter result by $\bar{\theta}$, and subtracting it from the former, which gives

$$\sum_x (\theta - \bar{\theta}) p'_x = 1. \quad (10)$$

The expression on the left-hand side is bounded from above by the Cauchy-Schwarz inequality,

$$1 \leq \left[\sum_x (\theta - \bar{\theta})^2 p_x \right] \left[\sum_x \frac{(p'_x)^2}{p_x} \right] = (\Delta\theta)^2 F, \quad (11)$$

so finally, we get

$$(\Delta\theta)^2 \geq 1/F, \quad (12)$$

which is the Cramer-Rao lower bound on the variance of an unbiased estimator. Of course, to achieve this resolution it is necessary to register a large number of particles n . As follows from the expansion of the likelihood, the performance of a ML estimator improves with increasing number of particles as $1/(nF)$. Hence the Fisher information gives the ultimate resolution corresponding to a single "average" particle from the bunch of registered events¹⁶.

III. FISHER INFORMATION OF INTERFERENCE PATTERNS

Let us apply the theory to the interference pattern behind the slit. It is easy task to calculate all the respective quantities

$$(\Delta\mu)^2 = \left(\frac{a}{2\hbar}\right)^2 (\Delta p_x)^2, \quad (13)$$

$$F = \frac{4}{\pi} \int d\mu \left(\frac{d}{d\mu} \text{sinc } \mu\right)^2 = \frac{4}{3}, \quad (14)$$

$$(\Delta x)^2 = \frac{a^2}{12}. \quad (15)$$

What is really intriguing, the Cramer-Rao inequality (12) reproduces exactly the expected Heisenberg uncertainty relations for impulse and position of the particle going

through the slit. Indeed, plugging the uncertainties (13), (14), and (15) into Eq. (12) the resulting uncertainty relation reads

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}. \quad (16)$$

Remarkably, the state of particle behind the slit meets the equality sign here being the minimum uncertainty state. The price paid for this interpretation is the reinterpretation of the measurement of the impulse. On the contrary to the single detection case, here the identity of separate particles is disregarded. What we observe is the impulse of the "center of mass" of the bunch of particles rather than the impulse of each particle separately.

But there are still some other remarkable differences in interpretation. Provided that accuracy is related to n -particle signal (for example, assuming n -particle absorption process), the accuracy improves according to the distribution (6) n times and the same effect appears in the Cramer-Rao inequalities (12). According to the standard interpretation, this improvement is viewed as the effect of a rescaling of the de Brogli wavelength due to the n times greater mass of interfering "quasi" particles^{6,7,8}. The improved accuracy is consequently referred to as a measurement beyond the Heisenberg limit, which of course seems to be problematic in the view of the above mentioned arguments.

Another point is also intriguing: The standard exposition of the Heisenberg uncertainty relations relies on the notion of measurable quantities. Students are taught that such observables correspond to hermitian operators and that the accuracy of such observations can be described by variances. Since both the variances of position and impulse appear in the standard Heisenberg uncertainty relations, it may seem at the first glance that that uncertainty relations limit the simultaneous (inaccurate) measurement of non-commuting observables. This is certainly not true. Heisenberg uncertainty relations express merely a necessary condition obeyed by any wave function.

The problem of simultaneous measurement of non-commuting observables is a more involved problem and is therefore beyond the exposition in the undergraduate course. It was first discussed by Arthurs and Kelly¹⁷ for position and impulse observables, and the answer relies upon the notion of generalized measurements. They showed that in the case of a simultaneous measurement of position and momentum, the product of the uncertainties becomes two times larger compared to the standard Heisenberg uncertainty relation. It is interesting to note that there is no danger of similar misinterpretation within the information theory. Particularly, the Cramer-Rao inequality for the estimation of a single variable (12) involves a single variance only. Hence the formulation anticipates the measurement of single parameter only (impulse). The variance of the position was introduced in order to establish the link to the standard Heisenberg uncertainty principle. Notice, however, that Cramer-Rao

inequalities may be easily extended to higher dimensions, and particularly, the Cramer-Rao inequality for simultaneous detection of impulse and position will reveal two times higher right-hand side¹⁷.

There is a simple relationship between the Fisher information and variances of complementary variables. Therefore, the Cramer-Rao inequalities imply the standard uncertainty relation¹⁸. Considering the momentum representation,

$$\frac{\partial}{\partial p}\psi(p) = \langle p | \frac{\hat{X}}{i\hbar} | \psi \rangle, \quad (17)$$

$$\frac{\partial}{\partial p}\psi(p)^* = -\langle \psi | \frac{\hat{X}}{i\hbar} | p \rangle, \quad (18)$$

$$\psi(p) = \langle p | \psi \rangle, \quad \psi(p)^* = \langle \psi | p \rangle, \quad (19)$$

the Fisher information may be rewritten to the form

$$F_p = \int d\mu \frac{[\psi(\mu)^* \psi(\mu) + \psi(\mu)' \psi(\mu)^*]^2}{\psi(\mu)\psi^*(\mu)} \quad (20)$$

$$= \int d\mu \frac{1}{\psi(\mu)\psi^*(\mu)} \langle \mu | \left[\frac{\Delta \hat{X}}{i\hbar}, |\psi\rangle\langle\psi| \right] | \mu \rangle^2. \quad (21)$$

Using the identity

$$|\langle \hat{A}\hat{B} \rangle|^2 = \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 + \frac{1}{4} |\langle \{\hat{A}, \hat{B}\} \rangle|^2,$$

we get

$$F_p = \frac{4}{\hbar^2} \langle \psi | (\Delta X)^2 | \psi \rangle \quad (22)$$

$$- \int d\mu \psi(\mu)\psi^*(\mu) \left[\frac{\partial}{\partial \mu} \arg(\psi(\mu)) + \frac{1}{\hbar} \bar{X} \right]^2,$$

where $\bar{X} = \langle \psi | \hat{X} | \psi \rangle$. That is why the Cramer-Rao inequality is stronger than the Heisenberg uncertainty relation, since always

$$(\Delta p)^2 \geq \frac{1}{F_p} \geq \frac{\hbar^2}{4(\Delta x)^2}. \quad (23)$$

They will coincide whenever the phase of the wave function is related to the mean value of position by the relation

$$\frac{\partial}{\partial \mu} \arg(\psi(\mu)) = -\frac{1}{\hbar} \bar{X}, \quad (24)$$

which means that the phase of the wave function of minimum uncertainty states in p -representation exhibits a linear dependence on the momentum, $\arg \psi(\mu) = \alpha + \beta\mu$, and, in particular, it must be constant if $\bar{X} = 0$. This condition gives a wide class of minimum uncertainty states for the informatic uncertainty relation on the slit.

IV. QUANTUM INTERFEROMETRY

The analysis of the diffraction on the slit from the information point of view may serve not merely as an alternative interpretation of this experiment but it could also

be used for the evaluation of the performance of interferometric techniques. The variance of the detected signal is crucial for the maximal attainable resolution. This can be seen from the following simple linearized theory, frequently used in various considerations. Let us assume the measurement of a generic operator A . This measurement should reveal the value of an unknown parameter $\bar{\theta}$ appearing in a unitary transformation $U(\bar{\theta})$ applied to a quantum state $|\psi\rangle$: $|\psi(\bar{\theta})\rangle = U(\bar{\theta})|\psi\rangle$. Provided that the average value of the θ -dependent signal $\langle A(\bar{\theta}) \rangle$ is detected, the desired value of the parameter may be inferred by means of an estimator θ . The estimated value is usually uncertain since the signal itself fluctuates with the variance given by $(\Delta A)^2 = (\langle A^2 \rangle - \langle A \rangle^2)$. Symbol $(\Delta)^2$ will be reserved for variances, whereas symbol δ will denote an error obtained by means of a linearized theory,

$$\delta\theta \approx \frac{\Delta A}{\langle A(\bar{\theta})' \rangle}, \quad (25)$$

where prime denotes a derivative with respect to the parameter. For its simplicity, the latter measure of error is frequently used in interferometry¹⁹. However, one should keep in mind that in some cases the two quantities can be significantly different. Let us evaluate a simple realistic model of the phase detection by means of a Mach-Zehnder interferometer, see Fig. 2. Formally, this device can be described by Lie algebra $SU(2)$, the correspondence being provided by the Schwinger representation of the angular momentum operators,

$$J_1 = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad (26)$$

$$J_2 = \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1),$$

$$J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2).$$

All these operators commute with the total number of particles $N = a_1^\dagger a_1 + a_2^\dagger a_2$ and satisfy commutation relations of angular momentum observables $[J_i, J_j] = i\epsilon_{ijk} J_k$. Provided that a difference of particle numbers on the output ports will be registered, the measurement is represented by J_3 operator. Before reaching the detectors, the input signal undergoes subsequent transformations on two beam-splitters and phase shifter. Considering a symmetrical beam splitter and merger, both introduce the phase shift of $\pi/2$ for the reflected signal, their action on the input state of light is described by the unitary operator $e^{-i\pi J_1}$. Phase shifter transforms the state according to the unitary transformation $e^{-i(\phi_2 - \phi_1)J_3}$. Consequently, the quantum state at the output carries the information about phase shift $\phi = \phi_2 - \phi_1$. The total transformation induced by the Mach-Zehnder interferometer then reads,

$$U(\phi) = e^{i\pi/2 J_1} e^{-i\phi J_3} e^{-i\pi/2 J_1} = e^{-i\phi J_2}, \quad (27)$$

$$|out\rangle = U(\phi)|in\rangle, \quad (28)$$

$$U^\dagger(\phi)J_3U(\phi) = -\sin\phi J_1 + \cos\phi J_3. \quad (29)$$

Assume now that the input ports are fed by n_1 and n_2 particles, respectively. Adopting the standard notation of eigenstates $|j, m\rangle$ of the operators J^2 and J_3 , the input state simply reads $|in\rangle = |j, m\rangle$, where $j = (n_1 + n_2)/2$ and $m = (n_1 - n_2)/2$. The output of the measurement can either be characterized by phase dependent moments

$$\langle J_3 \rangle = m \cos \phi, \quad (30)$$

$$\langle J_3^2 \rangle = m^2 \cos^2 \phi + 1/2[j(j+1) - m^2] \sin^2 \phi, \quad (31)$$

or, more completely, by phase dependent output statistics,

$$p_k(\phi) = |\langle j, k | U(\phi) | j, m \rangle|^2, \quad (32)$$

sampled by the measurement. Such a measurement is obviously complete $\sum_k p_k(\phi) = 1$. In the following, the description will further be simplified considering only phase shifts near the working point $\phi = 0$, where the ultimate limit of the accuracy of the phase-shift measurement can be evaluated most easily. The Fisher information associated with the phase estimation in this case reads,

$$F_0 = \sum_k \frac{[p'_k(0)]^2}{p_k(0)}, \quad (33)$$

Notice that all probabilities $p_k(\phi) = |\langle j, k | e^{-i\phi J_2} | j, m \rangle|^2$, evaluated at $\phi = 0$ vanish except when $k = m$. In the former case, this leads to an indefinite expression of the type $0/0$ under the summation sign in Eq. (33). Evaluating them with the help of the L'Hospital rule we obtain,

$$F_0 = \sum_{k \neq m} \frac{2p'_k p''_k}{p'_k} = 2 \sum_{k \neq m} p''_k = -2p''_m, \quad (34)$$

since $\sum p_k = 1$. Consequently, for the input state $|j, m\rangle$ and true phase $\phi = 0$, we found the Fisher information to be

$$F_0 = 4\langle j, m | J_2^2 | j, m \rangle = 2[j(j+1) - m^2]. \quad (35)$$

At the same time the phase error could be estimated by means of the simple linearized theory as

$$(\delta\phi)^2 = \frac{(\Delta J_3)^2}{(J_3(\phi)')^2} = \frac{j(j+1) - m^2}{2m^2}. \quad (36)$$

Now assume two cases of special interest related to the classical and quantum regimes. Provided that the interferometer is operated in the usual (classical) manner with the light entering one input port only, $|in\rangle = |j = \frac{N}{2}, m = \frac{N}{2}\rangle$, the phase error given by the linearized theory reads $\delta\phi = 1/\sqrt{N}$. Significantly, the ultimate variance predicted by the Cramer-Rao inequalities gives the same value of $\Delta\phi = 1/F_0 = 1/\sqrt{N}$. This regime is usually referred to as the standard limit of phase measurements.

Intriguing situation appears when both the input ports of the interferometer are fed by the signal with an equal

number of particles $N/2$. In this case we have state $|j = \frac{N}{2}, m = 0\rangle$ at the input and the situation becomes diametrically different. Here, the simple linearized theory fails to provide an error estimate because $\delta\phi = 0/0$. However, the information approach can still be used and the Fisher information predicts the ultimate phase resolution of

$$\Delta\phi = \sqrt{2}/N. \quad (37)$$

This regime is referred to as the quantum limit of phase measurements²⁰

Let us clarify some experimental aspects of this quantum resolution regime²¹. The probability distribution of the detected signal can be approximated for large values of j by

$$\mathcal{P}(\phi|m) = p_m(\phi) \approx J_m^2(\phi j), \quad (38)$$

where J_m denotes the Bessel function. According to the Bayes principle this expression also provides the posterior phase distribution conditioned on the detected photon number difference m . Obviously, the most accurate detection of the phase shift is expected for a carefully balanced interferometer ($\phi = 0$) when input state is transmitted without any change $m = 0$. However, an experimentalist is facing an inverse problem. Provided that the value $m = 0$ was registered, one cannot be sure whether the true value of the phase shift had really been set to zero. Of course, $m = 0$ could be detected with some probability for other phase shifts as well. There is always some uncertainty about the estimated parameter. As a proper measure of performance, the width of the respective posterior phase distribution can be adopted. Accepting the same heuristic argumentation as in the case of the diffraction on the slit, the first minimum of the Bessel function could be used to define the width. Indeed, the resolution obtained in this way is of the order of $(\delta\phi)^2 \approx 1/j^2$; $N = 2j$ which resembles the quantum phase resolution beating the standard noise limit of phase²². However, neither in the case of diffraction, nor here the distance from the center to the first minimum of the probability distribution is a plausible measure of error in quantum theory. One should realize that distribution (38) is not even square integrable in the limit of $j \rightarrow \infty$. Its variance can be evaluated assuming the phase window π :

$$(\Delta\phi)^2 = \left(\frac{1}{j}\right)^2 \frac{\int_0^{j\pi/2} dx J_0^2(x) x^2}{\int_0^{j\pi/2} dx J_0^2(x)} \propto 1/\ln j. \quad (39)$$

Here the Bessel function was approximated by its asymptotic expansion in the last step, since the heavy tails of the distribution yield a dominant contribution. What does this mathematical expression mean? Provided that the true phase is estimated from the result of a single shot measurement, the result is rather uncertain²³. In particular, such measurement is much worse compared to the interferometer operated in the usual manner (with single illuminated input port). However, our scheme still posses

some potential for an improvement. Let us imagine that our single-shot measurement is repeated. Provided the most optimistic result $m = 0$ is detected again, the posterior distribution will now become

$$P(\phi|m = 0; m = 0) \approx J_0^4(\phi j), \quad (40)$$

where $\{m = 0; m = 0\}$ represents accumulated phase-sensitive data. The procedure of data accumulation can be repeated until the optimal resolution is reached²¹. As can be shown, the accuracy predicted by the Fisher information is achieved provided that the single shot detections are repeated n times, $n \geq 4$. For instance, in the most optimistic case when $\{m = 0; m = 0; m = 0; \dots m = 0\}_n$ was detected, the variance of the posterior phase distribution is $(\Delta\phi)^2 = 1/(nj^2)$. Since the total number of particles used in the experiment is $N_{tot} = Nn$, the phase resolution scales with the total energy used as

$$(\Delta\phi)^2 = 4n/N_{tot}^2, \quad n \geq 4. \quad (41)$$

As can be seen the optimum regime is that with $n = 4$ repetitions. Notice however, that although the quantum phase resolution $\propto 1/N^2$ has been recovered, the optimal variance is still by a constant factor larger than the ultimate limit (37) predicted by the Fisher information. Indeed, the data accumulation that is needed to find the tiny sharp central peak of the phase distribution always tends to move the resolution from the quantum toward the classical regime; in the extreme case of $n = N_{tot}/2$ when the particles are injected into the interferometer in pairs we only get the classical resolution of $(\Delta\phi)^2 = 2/N_{tot}$. The fact that the ultimate Fisher limit cannot be attained is in accordance with the theory, since this ultimate limit need not be achieved by measurement. This analysis illustrates the application of the concept of Fisher information in practice.

The setup of quantum interference similar to quantum interferometry with equal number of particles beams has recently been proposed for beating the Rayleigh resolution limit in lithography⁷. The goal of the quantum lithography is to overcome the well-known limitation of classical optics and create a pattern in one and two dimensions whose resolution is below the diffraction limit. As any finite aperture cuts off the spatial frequencies higher than the inverse Rayleigh distance, the information about small details beyond this diffraction limit is inevitably lost. This may appear as detrimental, for instance, in nanotechnology. The proposal of quantum lithography is motivated by the same setup used in quantum interferometry. As has been demonstrated above, the quantum interference pattern can be created

by means of non-classical interference. The phase shift is induced by the differences of optical phases inside the interferometer. The argument of Bessel function depends on the energy of photon number correlated beams. The shape of pattern can be compressed beyond the standard diffraction limit, provided that the substrate is sensitive to N-photon absorption. This technological obstacle makes this technique rather tricky and speculative, at least, from the technological point of view. However, the argumentation based on Fisher information raises other questions.

The width of the distribution in the sense of the distance to the first minimum (Rayleigh criterion) can certainly be decreased. However, as the price, only a small portion of the whole probability (or energy) concentrates there. The remaining portion of the signal is spread out and contributes to the overall noise. This undesirable effect will obviously influence the possibility to shape the desired pattern due to the unavoidable fluctuations. For applications in lithography one needs the possibility to create and transmit high frequencies as well as the capability to point to a target.

V. CONCLUSIONS

We have formulated several arguments in favor of Fisher information and its applications to quantum problems. Fisher information provides the ultimate limitation for quantum measurements and as such also provides a nontrivial link between the theory of statistics and quantum theory. Since quantum theory is more "operational" than perhaps any other physical theory, this may yield new interesting insights into its fundamentals. From the pedagogical point of view, it is especially interesting to apply the Fisher information to simple thought experiments frequently discussed in elementary textbooks on quantum mechanics. Particularly, we have shown that the example of the diffraction on the slit can be interpreted as the uncertainty principle for the impulse and position of particles creating the diffraction pattern. Similar argumentation can be used for the description of the phase resolution of quantum interferometry.

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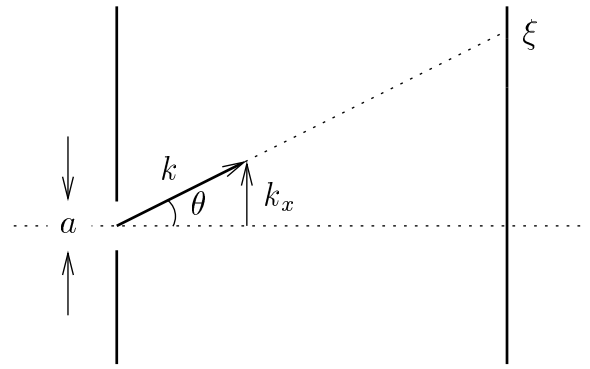


FIG. 1: Geometry of the diffraction on the slit.

RehacekFig1.eps

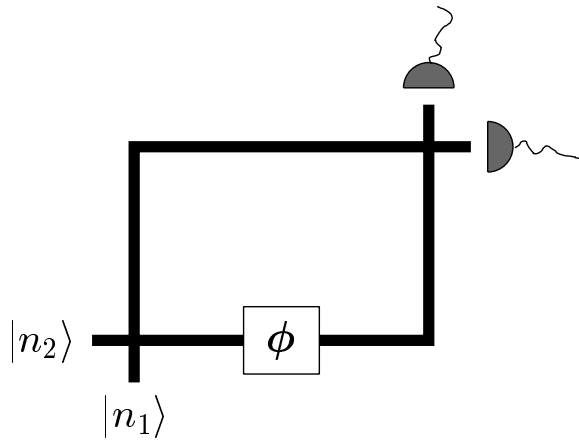


FIG. 2: Mach-Zehnder interferometer.

RehacekFig2.eps