Genuine multipartite entanglement conditions

Evgeny Shchukin
Peter van Loock

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Entanglement

- A bipartite separable state

$$\hat{\varrho} = \sum_k p_k \hat{\varrho}_k^{(1)} \otimes \hat{\varrho}_k^{(2)}, \quad \text{where} \quad p_k \geqslant 0, \ \sum_k p_k = 1$$

- Different notions of separability in multipartite case
  - In three-partite case there 4 kinds of entanglement — 1|23, 2|13, 3|12 and 1|2|3, which correspond to

$$\hat{\varrho} = \sum p_k \hat{\varrho}_k^{(1)} \otimes \hat{\varrho}_k^{(23)}$$
$$\hat{\varrho} = \sum p_k \hat{\varrho}_k^{(2)} \otimes \hat{\varrho}_k^{(13)}$$
$$\hat{\varrho} = \sum p_k \hat{\varrho}_k^{(3)} \otimes \hat{\varrho}_k^{(12)}$$
$$\hat{\varrho} = \sum p_k \hat{\varrho}_k^{(1)} \otimes \hat{\varrho}_k^{(2)} \otimes \hat{\varrho}_k^{(3)}$$

Entanglement

- In general, a kind of separability is a decomposition $\{I_1, \ldots, I_m\}$. In the previous case $1|23$-separability corresponds to $\{\{1\}, \{2, 3\}\}$ and $1|2|3$-separability to $\{\{1\}, \{2\}, \{3\}\}$

- For $n \geqslant 2$ there are $2^{n-1} - 1$ bipartitions and many more partitions into more parts

- A tripartite state is biseparable if it is a mixture

$$\hat{\varrho} = p_1 \hat{\varrho}_1 + p_2 \hat{\varrho}_2 + p_3 \hat{\varrho}_3$$

  where $\hat{\varrho}_1$ is $1|23$-separable, $\hat{\varrho}_2$ is $2|13$-separable and $\hat{\varrho}_3$ is $3|12$-separable

- In general, a state is biseparable if is a mixture of $2^{n-1} - 1$ state that are separable with respect to all possible bipartitions

Separability conditions

- A classical result[1]

$$\langle (\hat{x}_1 + \hat{x}_2)^2 + (\hat{p}_1 - \hat{p}_2)^2 \rangle \geqslant 2$$

or, in a more general form

$$\left\langle \left(a\hat{x}_1 + \frac{1}{a}\hat{x}_2 \right)^2 + \left(a\hat{p}_1 - \frac{1}{a}\hat{p}_2 \right)^2 \right\rangle \geqslant \begin{cases} a^2 + \frac{1}{a^2} & \text{for sep.} \\ \left| a^2 - \frac{1}{a^2} \right| & \text{for quant.} \end{cases}$$

- There are two boundaries — one for quantumness and one separability

- Many boundaries in a multipartite case

[1] L.-M. Duan, G. Giedke, J. I. Cirac and P. Zoller *Inseparability Criterion for Continuous Variable Systems*, Phys. Rev. Lett. **84**, 2722 (2000)

Conditions for bipartite separability

- The inequality
$$\langle (\hat{x}_1 + \hat{x}_2)^2 + (\hat{p}_1 - \hat{p}_2)^2 \rangle \geqslant 2$$
is equivalent to the inequality
$$\langle (\hat{a}^\dagger + \hat{b})(\hat{a} + \hat{b}^\dagger) \rangle \geqslant 1 \tag{1}$$
- If $\hat{A}$ and $\hat{B}$ act on different parts, and $[\hat{B}, \hat{B}^\dagger] = c > 0$, then[2]
$$\langle (\hat{A}^\dagger + \hat{B})^n (\hat{A} + \hat{B}^\dagger)^n \rangle \geqslant c^n n!$$
- For $\hat{A} = \hat{a}$, $\hat{B} = \hat{b}$ (so that $c = 1$) and $n = 1$ we get Eq. (1)
- For $\hat{A} = \hat{a}$, $\hat{B} = \hat{b}$ and $n \geqslant 1$ we have[3]
$$\langle (\hat{a}^\dagger + \hat{b})^n (\hat{a} + \hat{b}^\dagger)^n \rangle \geqslant n!$$
- We can use multipartite $\hat{A}$ and $\hat{B}$!

[2] E.Shchukin and P. van Loock *Tripartite separability conditions exponentially violated by Gaussian states*, Phys. Rev. A **90**, 012334 (2014)
[3] H. Nha, S.-Y. Lee, S.-W. Ji and M. S. Kim *Efficient entanglement criteria beyond Gaussian limits using Gaussian measurements*, Phys. Rev. Lett. **108**, 030503 (2012)

Conditions for tripartite separability

- Let us take $\hat{A} = \hat{a}$ and $\hat{B} = \hat{b} + \hat{c}$, and denote

$$\mathscr{A}_{1|23}^{(n)} = \frac{1}{n!}\langle(\hat{a}^{\dagger} + \hat{b} + \hat{c})^{n}(\hat{a} + \hat{b}^{\dagger} + \hat{c}^{\dagger})^{n}\rangle$$

Then we have the inequality

$$\mathscr{A}_{1|23}^{(n)} \geqslant \begin{cases} 2^{n} & \text{for all } 1|23\text{-separable states} \\ 1 & \text{for all states} \end{cases}$$

- The tripartite pure Gaussian state with wave function

$$\psi(\mathbf{x}) = \frac{1}{\sqrt[4]{\pi^{3}}} e^{-\frac{1}{2}\mathbf{x}^{T}A\mathbf{x}}, \qquad A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

is the eigenstate of $\hat{a}^{\dagger} + \hat{b} + \hat{c}$ with eigenvalue zero and for this state we have $\mathscr{A}_{1|23}^{(n)} = 1$

Conditions for tripartite separability

- Let us introduce the symmetrized sum

$$\mathscr{S}^{(n)} = \frac{1}{3}(\mathscr{A}^{(n)}_{1|23} + \mathscr{A}^{(n)}_{2|13} + \mathscr{A}^{(n)}_{3|12})$$

Then we have the inequality

$$\mathscr{S}^{(n)} \geqslant \begin{cases} 2^n & \text{for fully separable states} \\ \frac{2^n+3}{3} & \text{for biseparable states} \\ 1 & \text{for all states} \end{cases}$$

- The inequality $\mathscr{S}^{(n)} > 1$ is always strict and tight
- For the tripartite pure Gaussian state with wave function

$$\psi(\mathbf{x}) = \sqrt[4]{\frac{\det A_\xi}{\pi^3}} e^{-\frac{1}{2}\mathbf{x}^T A_\xi \mathbf{x}}, \quad A_\xi = \begin{pmatrix} 1 & \xi & \xi \\ \xi & 1 & \xi \\ \xi & \xi & 1 \end{pmatrix}$$

we have $\mathscr{S}^{(n)} \to 1$ when $\xi \to 1$. Violation grows as 2^n!

Violation

- For the GHZ/W state $|\varphi_a\rangle$, which are pure Gaussian states with

$$A_a = \begin{pmatrix} a & e & e \\ e & a & e \\ e & e & a \end{pmatrix}, \quad e = \frac{a^2 - 1 - \sqrt{(a^2-1)(9a^2-1)}}{4a}$$

  for $a = 1.5$ the violation grows as 1.16^n.
- Let us introduce

$$\tilde{\mathscr{A}}^{(n)}_{1|23} = \frac{1}{n!}\langle(-\hat{a}^\dagger + \hat{b} + \hat{c})^n(-\hat{a} + \hat{b}^\dagger + \hat{c}^\dagger)^n\rangle$$

  then we can define $\tilde{\mathscr{S}}^{(n)}$ in the same way as $\mathscr{S}^{(n)}$ and this new quantity will satisfy the same inequalities
- If a pure Gaussian state with matrix A violates the inequalities for $\mathscr{S}^{(n)}$ for some n then the Gaussian state with matrix A^{-1} violates the inequalities for $\tilde{\mathscr{S}}^{(n)}$ for the same n

Conditions for tripartite separability

- In terms of position and momentum operators our inequalities for $n = 1$ read as follows. Let us define

$$\begin{aligned}\mathscr{T} &= 3\langle(\hat{x}_a + \hat{x}_b + \hat{x}_c)^2\rangle + \langle(-\hat{p}_a + \hat{p}_b + \hat{p}_c)^2\rangle \\ &+ \langle(\hat{p}_a - \hat{p}_b + \hat{p}_c)^2\rangle + \langle(\hat{p}_a + \hat{p}_b - \hat{p}_c)^2\rangle\end{aligned}$$

  Then the inequalities for $\mathscr{S}^{(1)}$ are equivalent to

$$\mathscr{T} \geqslant \begin{cases} 9 & \text{for fully separable states} \\ 5 & \text{for biseparable states} \\ 3 & \text{for all states} \end{cases}$$

- These inequalities can be proven without PT!
  The key points are the inequality $\langle\hat{X}^2 + \hat{Y}^2\rangle \geqslant |\langle[\hat{X}, \hat{Y}]\rangle|$ for arbitrary observables $\hat{X}$ and $\hat{Y}$, and the relation for the commutator $[\hat{x}_a + \hat{x}_b + \hat{x}_c, -\hat{p}_a + \hat{p}_b + \hat{p}_c] = i$ (and the two similar ones)

Second-order conditions

- Let M be a real, symmetric, positive-definite $2n \times 2n$ matrix

$$\mathscr{G} = \langle \mathbf{r}^{\mathrm{T}} M \mathbf{r} \rangle$$

  where $\mathbf{r} = (\mathbf{x}, \mathbf{p})$

- Usually block-diagonal matrices are considered

$$M = \begin{pmatrix} X & 0 \\ 0 & P \end{pmatrix}$$

  In this case

$$\mathscr{G} = \mathrm{Tr}(X\gamma_{xx}) + \mathrm{Tr}(P\gamma_{pp})$$

  where $\gamma_{xx} = (\langle \hat{x}_i \hat{x}_j \rangle)_{i,j=1}^{n}$ and $\gamma_{pp} = (\langle \hat{p}_i \hat{p}_j \rangle)_{i,j=1}^{n}$

- Replacing $\psi(\mathbf{x})$ by $\psi(\mathbf{x} + \mathbf{x}_0)e^{-i(\mathbf{x},\mathbf{p}_0)}$, where $\mathbf{x}_0 = \langle \mathbf{x} \rangle$, $\mathbf{p}_0 = \langle \mathbf{p} \rangle$, we can also use $\Delta\mathbf{r}$ instead of $\mathbf{r}$

Second-order conditions

- The condition

$$\langle (\hat{x}_1 + \hat{x}_2)^2 + (\hat{p}_1 - \hat{p}_2)^2 \rangle \geqslant 2$$

  uses matrices

$$X = \begin{pmatrix} a^2 & 1 \\ 1 & a^{-2} \end{pmatrix} = \begin{pmatrix} a \\ a^{-1} \end{pmatrix} \begin{pmatrix} a & a^{-1} \end{pmatrix}, \quad P = \begin{pmatrix} a^2 & -1 \\ -1 & a^{-2} \end{pmatrix} = \begin{pmatrix} a \\ -a^{-1} \end{pmatrix} \begin{pmatrix} a & -a^{-1} \end{pmatrix}$$

- The work[4] general rank-one matrices

$$X = \mathbf{h}\mathbf{h}^{\mathrm{T}}, \quad P = \mathbf{g}\mathbf{g}^{\mathrm{T}}$$

  where $\mathbf{h}$ and $\mathbf{g}$ are arbitrary real non-zero n-vectors

- The work[5] uses general matrices X and P

[4] P. van Loock and A. Furusawa *Detecting genuine multipartite continuous-variable entanglement*, Phys. Rev. A **67**, 052315 (2003)
[5] S. Gerke et al *Full multipartite entanglement of frequency-comb Gaussian states*, Phys. Rev. Lett. **114**, 050501 (2015)

Convex optimization

- *... the discovery that convex optimization problems (beyond least-squares and linear programs) are more prevalent in practice than was previously thought. Since 1990 many applications have been discovered in areas such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics, and finance. Convex optimization has also found wide application in combinatorial optimization and global optimization, where it is used to find bounds on the optimal value, as well as approximate solutions. We believe that many other applications of convex optimization are still waiting to be discovered.*

- *There are great advantages to recognizing or formulating a problem as a convex optimization problem. The most basic advantage is that the problem can then be solved, very reliably and efficiently, using interior-point methods or other special methods for convex optimization. These solution methods are reliable enough to be embedded in a computer-aided design or analysis tool, or even a real-time reactive or automatic control system.*[6]

[6] S. Boyd and L. Vandenberghe *Convex optimization*, Cambridge University Press, 2009

The minimal value

- What is the minimal of $\langle \mathbf{r}^{\mathrm{T}} M \mathbf{r} \rangle$ over all quantum states?

- The Williamson theorem[7] states that there is a symplectic matrix S such that

$$S^{\mathrm{T}} M S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

- Every symplectic transform is implementable as a unitary transformation[8] so that

$$\langle \mathbf{r}^{\mathrm{T}} M \mathbf{r} \rangle = \left\langle \mathbf{r}^{\mathrm{T}} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \mathbf{r} \right\rangle'$$

  and thus the minimal value of $\langle \mathbf{r}^{\mathrm{T}} M \mathbf{r} \rangle$ is equal to $\mathrm{Tr}\, \Lambda$

- For block-diagonal M we have $\mathrm{Tr}\, \Lambda = \mathrm{Tr} \sqrt{\sqrt{X} P \sqrt{X}}$

[7] M. de Gosson *Symplectic geometry and quantum mechanics*, Birkhäuser Verlag, 2000

[8] R. Simon, E. C. G. Sudarshan, and N. Mukunda, *Gaussian pure states in quantum mechanics and the symplectic group*, Phys. Rev. A 37, 3028 (1988)

Trace inequalities

- We have the tight inequality

$$\mathscr{G} = \mathrm{Tr}(X\gamma_{xx}) + \mathrm{Tr}(P\gamma_{pp}) \geqslant \mathrm{Tr}\sqrt{\sqrt{X}P\sqrt{X}}$$

- For a pure state with wave function $\psi(\mathbf{x})$ we have

$$\mathscr{G} = \int \Big((\mathbf{x}^{\mathrm{T}} X \mathbf{x}) |\psi(\mathbf{x})|^2 + (\nabla\psi^*(\mathbf{x}))^{\mathrm{T}} P (\nabla\psi(\mathbf{x})) \Big) d\mathbf{x}$$

- If $\psi(\mathbf{x}) = f(\mathbf{x}) e^{i\varphi(\mathbf{x})}$, where $f(\mathbf{x})$ is real, then

$$\int (\nabla\psi^*)^{\mathrm{T}} P (\nabla\psi)\, d\mathbf{x} = \int \Big((\nabla f)^{\mathrm{T}} P (\nabla f) + f^2 (\nabla\varphi)^{\mathrm{T}} P (\nabla\varphi) \Big) d\mathbf{x}$$

- We see that $\mathscr{G}(\psi) \geqslant \mathscr{G}(f) \Rightarrow$ real wave functions are sufficient

Trace inequalities

- For real wave function $f(\mathbf{x})$ we have

$$\mathscr{G} = \int \Big((\mathbf{x}^{\mathrm{T}} X \mathbf{x}) f(\mathbf{x})^2 + (\nabla f(\mathbf{x}))^{\mathrm{T}} P (\nabla f(\mathbf{x})) \Big)\, d\mathbf{x}$$

- Introduce $\mathbf{u}(\mathbf{x}) = \sqrt{X} f(\mathbf{x})\mathbf{x}$ and $\mathbf{v}(\mathbf{x}) = \sqrt{P}(\nabla f)(\mathbf{x})$, then

$$\mathscr{G} = \int \Big(\|\mathbf{u}(\mathbf{x})\|^2 + \|\mathbf{v}(\mathbf{x})\|^2 \Big)\, d\mathbf{x}$$

- Applying Cauchy-Schwarz inequality, we get

$$\mathscr{G} \geqslant 2 \left| \int (\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}))\, d\mathbf{x} \right| = 2 \left| \int f(\mathbf{x}) \mathbf{x}^{\mathrm{T}} \sqrt{X} \sqrt{P} (\nabla f)(\mathbf{x})\, d\mathbf{x} \right|$$

- The latter expression is easily shown to be equal to $\mathrm{Tr}(\sqrt{X}\sqrt{P})$

Trace inequalities

- We obtain the inequality

$$\mathrm{Tr}\sqrt{\sqrt{X}P\sqrt{X}} \geqslant \mathrm{Tr}(\sqrt{X}\sqrt{P}) = \mathrm{Tr}(\sqrt[4]{X}\sqrt{P}\sqrt[4]{X}) \qquad (2)$$

- Araki-Lieb-Thirring trace inequalities[9]

$$\mathrm{Tr}(A^{1/2}BA^{1/2})^{rq} \geqslant \mathrm{Tr}(A^{r/2}B^{r}A^{r/2})^{q}$$

  for positive definite matrices A and B and, $q \geqslant 0$ and $\leqslant r \leqslant 1$

- For $q = 1$ and $r = 1/2$ we get the inequality (2)

[9] E. H. Lieb and W. E. Thirring *Bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett. **35**, 687 (1975)
E. H. Lieb and W. E. Thirring *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, 269–303 (1976)
H. Araki *On an inequality of Lieb and Thirring*, Lett. Math. Phys. **19**, 167 (1990)

Separability condition

- If $f(\mathbf{x})$ is a real wave function of the form $f(\mathbf{x}) = g(\mathbf{x}')h(\mathbf{x}'')$ and parts i and j are separable than $\langle \hat{p}_i \hat{p}_j \rangle = 0$. The values $\langle \hat{x}_i \hat{x}_j \rangle$ factorize: $\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle$

- Replacing $f(\mathbf{x})$ by $f(\mathbf{x} + \mathbf{x}_0)$, where $\mathbf{x}_0 = \langle \mathbf{x} \rangle$, $\mathscr{G}$ gets replaced by $\mathscr{G} - \langle \mathbf{x} \rangle^{\mathrm{T}} \mathrm{X} \langle \mathbf{x} \rangle \leqslant \mathscr{G}$

- To minimize $\mathscr{G}$ we can assume that $\langle \mathbf{x} \rangle = 0$ and thus that $\langle \hat{x}_i \hat{x}_j \rangle = 0$ if parts i and j are separable

- We have

$$
\begin{aligned}
\mathscr{G} &= \mathrm{Tr}(X\gamma_{xx}) + \mathrm{Tr}(P\gamma_{pp}) \\
&= \mathrm{Tr}(X'(\mathbf{u})\gamma_{xx}) + \mathrm{Tr}(P'(\mathbf{v})\gamma_{pp})
\end{aligned}
$$

Separability condition

- For example: For $2|134$ and $1|2|34$-separability the matrix $X'(\mathbf{u})$ reads as

$$\begin{pmatrix} X_{11} & u_1 & X_{13} & X_{14} \\ u_1 & X_{22} & u_2 & u_3 \\ X_{13} & u_2 & X_{33} & X_{34} \\ X_{14} & u_3 & X_{34} & X_{44} \end{pmatrix}, \quad \begin{pmatrix} X_{11} & u_1 & u_2 & u_3 \\ u_1 & X_{22} & u_4 & u_5 \\ u_2 & u_4 & X_{33} & X_{34} \\ u_3 & u_5 & X_{34} & X_{44} \end{pmatrix}$$

The matrix $P'(\mathbf{v})$ is constructed in the same way.

- If a state is separable than

$$\mathcal{G} \geqslant \max_{\mathbf{u},\mathbf{v}} \sqrt{\sqrt{X'(\mathbf{u})}P'(\mathbf{v})\sqrt{X'(\mathbf{u})}}$$

- To each kind of separability corresponds its own optimization problem

Entanglement condition

- We have to minimize the function

$$\mathscr{E}(X,P) = \mathscr{G}(X,P) - \max_{\mathbf{u},\mathbf{v}} \sqrt{\sqrt{X'(\mathbf{u})}P'(\mathbf{v})\sqrt{X'(\mathbf{u})}}$$

- Errors of the measurements should be taken into account!
- The right function to minimize is

$$\mathscr{E}(X,P) = \mathscr{G}(X,P) + s\,\sigma(X,P) - \max_{\mathbf{u},\mathbf{v}} \sqrt{\sqrt{X'(\mathbf{u})}P'(\mathbf{v})\sqrt{X'(\mathbf{u})}}$$

where

$$\sigma(X,P) = \sqrt{\sum_{i,j}^{n} \sigma_{xx,ij}^2 X_{ij}^2 + \sum_{i,j}^{n} \sigma_{pp,ij}^2 P_{ij}^2}$$

Entanglement condition

- The number s determines the probability $\mathbf{P}(s)$ of the right result. If the measurements of the elements of the covariance matrix have Gaussian distribution then

$$\mathbf{P}(s) = \operatorname{erf}\left(\frac{s}{2}\right)$$

  Form example, for $s = 3$ (so-called "three sigma rule") $\mathbf{P} = 0.997$.

- The function $\mathscr{E}(X, P)$ is convex!

- Due to uniformness, $\mathscr{E}(\lambda X, \lambda P) = \lambda \mathscr{E}(X, P)$ for $\lambda \geqslant 0$, it is better to consider an optimization problem with linear constraints

$$\min_{\mathrm{Tr}(X\gamma_{xx}+P\gamma_{pp})=C} \left(s\,\sigma(X, P) - \max_{\mathbf{u},\mathbf{v}} \sqrt{\sqrt{X'(\mathbf{u})}P'(\mathbf{v})\sqrt{X'(\mathbf{u})}}\right) < -C.$$

Analytical condition

- Let us consider the quantity

$$\mathscr{G}_n = \sum_{1\leq i<j\leq n} \langle(\hat{x}_i + \hat{x}_j)^2 + (\hat{p}_i - \hat{p}_j)^2\rangle.$$

- It is the general quantity $\mathscr{G}$ with

$$X = \begin{pmatrix} n-1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & n-1 \end{pmatrix}, \quad P = \begin{pmatrix} n-1 & \dots & -1 \\ \dots & \dots & \dots \\ -1 & \dots & n-1 \end{pmatrix}$$

- Quantumness bound

$$\mathscr{G}_n > (n-1)\sqrt{n(n-2)}$$

Analytical condition

- It can be shown that

$$\max_{\mathbf{u},\mathbf{v}} \sqrt{\sqrt{X'(\mathbf{u})}P'(\mathbf{v})\sqrt{X'(\mathbf{u})}} \geqslant (n-1)\sqrt{n(n-2)} + \frac{4k(n-k)}{\sqrt{n}(\sqrt{2n-2}+\sqrt{n-2})}$$

- If a state satisfies the inequality

$$\mathscr{G}_n < (n-1)\sqrt{n(n-2)} + \frac{4(n-1)}{\sqrt{n}(\sqrt{2n-2}+\sqrt{n-2})}$$

then it is genuine entangled

Analytical condition

n	2	3	4	5	6	7	8
q	0	3.46	8.48	15.49	24.49	35.49	48.49
a	2	5.00	10.03	17.06	26.07	37.08	50.09
b	2	5.46	10.89	18.26	27.59	38.89	52.17
f	2	6	12	20	30	42	56

Conclusion

- Conditions for genuine multipartite entanglement
- Exponential violation by pure tripartite Gaussian states
- Potential applicability to non-Gaussian states