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# Discrimination between non-perfectly known states

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## Abstract

Discrimination task is treated in the case of only partial prior information from measurements of unknown states. The construction of the optimal discrimination device and estimation of unknown states is performed simultaneously. A communication through a noisy quantum channel is formulated in terms of the proposed discrimination protocol. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Nonorthogonality of generic quantum states as a pure quantum mechanical phenomenon poses a fundamental obstacle to discrimination between them. Impossibility of perfect distinguishing between non-orthogonal quantum states limits the efficiency of many quantum information protocols, for example, communication through a noisy quantum channel. Historically, the first investigation of quantum discrimination was based on von Neumann measurements [1]. It determined the error rate bound of discrimination between two pure states. The general Bayesian approach to distinguishing between mixed states with non-equal prior probabilities based on minimization of cost function and positive operator valued measurements (POVM) was developed by Holevo and Hel-

strom [2,3]. In 1987 Ivanovic published a different model of two linear independent pure states discrimination without errors but with the occurrence of an inconclusive result [4]. Subsequently, the minimal probability of the inconclusive result was shown by Dieks and Peres [5,6] to be the overlap between the states being distinguished. Recently, discrimination techniques have been thoroughly studied, and analytical expressions for optimal POVMs for many quantum discrimination problems have been derived [7–14]. However, analytical results cannot be easily obtained for discrimination problems which incorporate many quantum states without a particular symmetry and numerical methods appear efficient [15,16]. Some discrimination devices have been realized experimentally [17–20].

So far distinguishability has been studied in the case of maximal information on the states to be distinguished. In this classical approach it is assumed that the sources  $H_i$  prepare physical systems, for ex-

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ample, electron spins or modes of light, in explicitly known quantum states  $\rho_i$ . Discrimination consists in distinguishing between these known states  $\rho_i$  or equivalently sources  $H_i$  (they are called hypotheses) having a physical system from one randomly chosen source. The formal solution of this problem is based on the construction of the optimal POVM that minimizes a convenient cost function, for example, the error rate or negative information [2,3]. The goal of this Letter is to develop a discrimination technique applicable to the case when only partial information on hypotheses  $H_i$  is available. Let us imagine that the quantum states  $\rho_i$  prepared by the sources  $H_i$  are not explicitly known but we are allowed to probe each source before discrimination. We can ask what quantum states are produced by these sources and what is the optimal POVM to distinguish between them. This scenario realizes, for example, in communication through an unknown distorting quantum channel. The hypotheses are explicitly known by transmitter party (Alice) but not by receiver party (Bob). In the calibration stage Alice sends physical systems in the quantum states  $\rho'_i$  from the sources  $H_i$  in given order down the channel while Bob receives sequence of distorted states  $\rho_i$ . He acquires some information (prior data) on hypothesis  $H_i$  through a generic measurement of states  $\rho_i$ . In communication stage, which follows, Alice sends a message with the help of her alphabet  $H_i$ . Transmission is successful if Bob is able to discriminate between hypothesis  $H_i$  or equivalently between quantum states  $\rho_i$  with the help of prior data.

The article is organized as follows. The standard discrimination is summarized in Section 2. Novel method of prior data manipulation for simultaneous estimation of the unknown states and design of the optimal discrimination POVM is proposed in Section 3. Two examples are shown in Section 4.

## 2. Standard approach

Let us consider the following discrimination protocol. Alice chooses in secret a quantum state from the set of two known quantum states  $|1\rangle, |2\rangle$  and sends it to Bob who must decide which state was chosen using a POVM measurement. For simplicity, but without the loss of generality, we will consider the same probability for choosing  $|1\rangle$  and  $|2\rangle$ . A natural measure of

Bob's failure is the error rate  $ER = (N_{12} + N_{21})/2N$ , where  $N$  is the number of trials,  $N_{12}$  is the number of Bob's wrong decisions  $|1\rangle$  when  $|2\rangle$  was true and  $N_{21}$  is the number of his wrong decisions  $|2\rangle$  when  $|1\rangle$  was true. Nonorthogonality of states implies the ultimate lower limit of the error rate—the Helstrom bound [3],

$$ER \geq \frac{1}{2} \left( 1 - \sqrt{1 - |\langle 1|2\rangle|^2} \right). \quad (1)$$

The optimal POVM measurement  $\Pi_1, \Pi_2$  for the discrimination between two mixed states  $\rho_1, \rho_2$  is the one that minimizes the error rate

$$ER = \frac{1}{2} (\text{Tr}[\Pi_1 \rho_2] + \text{Tr}[\Pi_2 \rho_1]) \quad (2)$$

subject to the constraint  $\Pi_1 + \Pi_2 = 1$ , or equivalently maximizes the functional

$$\mathbb{E} = \text{Tr}[\Pi_1 \rho_1] + \text{Tr}[\Pi_2 \rho_2] - \text{Tr}[\lambda(\Pi_1 + \Pi_2)], \quad (3)$$

where  $\Pi_{1,2}$  are now treated as independent variables. Variation of (3) subject to the positivity constraint on the operators  $\Pi_{1,2}$  gives the extremal equations for the optimal POVM elements and for the Lagrange operator multiplier  $\lambda$ ,

$$\begin{aligned} \rho_1 \Pi_1 &= \lambda \Pi_1, \\ \rho_2 \Pi_2 &= \lambda \Pi_2, \quad \rho_1 \Pi_1 + \rho_2 \Pi_2 = \lambda. \end{aligned} \quad (4)$$

For example, solving extremal equations (4) for two known pure states of qubit  $|\psi_{1,2}\rangle = \cos \alpha |+\rangle \pm \sin \alpha |-\rangle$  or corresponding mixed states

$$\rho_{1,2} = \begin{pmatrix} \cos^2 \alpha & \pm d \cos \alpha \sin \alpha \\ \pm d \cos \alpha \sin \alpha & \sin^2 \alpha \end{pmatrix}, \quad (5)$$

parametrized by  $\alpha \in [0, \pi/4]$  and  $d \in [0, 1]$ , yields the optimal POVM and ultimate error rate,

$$\begin{aligned} \Pi_{1,2} &= \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} = |\pm x\rangle \langle \pm x|, \\ ER &= \frac{1}{2} (1 - d \sin 2\alpha). \end{aligned} \quad (6)$$

## 3. Discrimination from partial information

In the previous section we assumed that Bob knows the states  $\rho_{1,2}$  explicitly and thus he can use this maximal prior information for designing the optimal discrimination POVM. However, in real communication through an unknown noisy channel Bob does not

know these states perfectly. Therefore, a calibration stage needs to be performed first. In this stage Alice transmits a given number of physical systems in state  $\rho'_1$  from the known source  $H_1$  through communication channel and Bob applies a prior POVM measurement  $\pi_k$  to the unknown received state  $\rho_1$ . He obtains real noisy data  $f_{1k}$  that approximate true probabilities  $p_{1k} = \text{Tr}[\rho_1 \pi_k]$ . The same calibration is done for source  $H_2$ . The prior information obtained by Bob during the calibration stage is equivalent to the explicit knowledge of the states  $\rho_{1,2}$  only in the non-physical case of tomographically complete POVM  $\pi_k$  and infinitely large number of transmitted states. The most simple and straightforward solution of this modified discrimination problem is obvious. Bob can estimate the states  $\rho_1, \rho_2$  from the prior data  $f_{1k}, f_{2k}$  and then solve the equations (4) in order to obtain the optimal POVM  $\Pi_{1,2}$ . Unfortunately, succession of these two steps does not represent the optimal strategy. If the data are not sufficient for accurate reconstruction of the states the deviations can be propagated to discrimination step. Another important problem arises from solution of the extremal equations derived from non-symmetric mixed states obtained as a result of some numerical state reconstruction technique [21–25].

Now we will formulate the modified discrimination protocol in the general case of  $I$  sources  $H_i, i = 1, \dots, I$ , and will show information-based solution of this problem. Let us start with  $I$  unknown mixed states  $\rho_i$  produced by the sources  $H_i$  and propagated through a noisy channel. We observe these states in the calibration stage by prior POVM  $\pi_k, k = 1, \dots, K$ , and obtain prior data  $f_{ik}, \sum_k f_{ik} = 1$ , that approximate the true probabilities

$$p_{ik} = \text{Tr}[\rho_i \pi_k]. \tag{7}$$

In the discrimination stage the states  $\rho_i$  are examined in virtual sense by designed discrimination POVM  $\Pi_j, j = 1, \dots, J$ . If we actually performed this measurement we would obtain frequencies  $F_{ij}, \sum_j F_{ij} = 1$ , sampling probabilities

$$P_{ij} = \text{Tr}[\rho_i \Pi_j]. \tag{8}$$

For simplicity, let us consider the same number of states  $\rho_i$  and designed POVM elements  $\Pi_j, I = J$ . In the case of more POVM elements than discriminated states the proposed method would generalize the

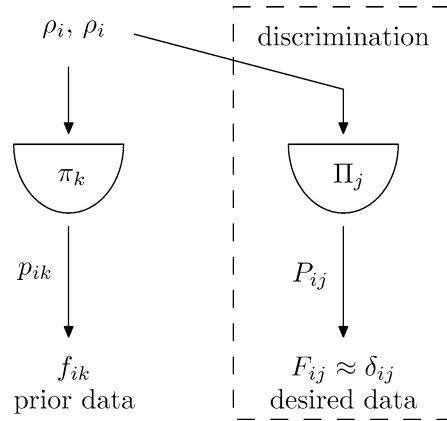


Fig. 1. Scheme of the experimental setup used for prior measurement of quantum states in calibration stage and design of the optimal discrimination POVM.

Ivanovic–Dieks–Peres [4–6] unambiguous discrimination or combination of both methods, not the pure Helstrom scheme. For optimal POVM the sum of diagonal frequencies  $\sum_i F_{ii}$  should be maximal and the error rate  $\sum_{ij} F_{ij} - \sum_i F_{ii}$  should be minimal. Thus the desired discrimination POVM should yield data  $F_{ij}$  as close to Kronecker’s delta  $\delta_{ij}$  as possible. This is schematically shown in Fig. 1.

In the case of the finite number of states used for calibration the equalities  $f_{ik} = p_{ik}$  do not hold exactly. Similarly the equalities  $F_{ij} = P_{ij} = \delta_{ij}$  are not satisfied because the perfect discrimination between non-orthogonal states is not possible. These equalities are satisfied only approximately and thus cannot be directly used for the estimation of the states  $\rho_i$  and design of the optimal POVM  $\Pi_j$ . Therefore, equality between data  $f_{ik}, F_{ij}$  and probabilities  $p_{ik}, P_{ij}$  must be handled in weaker—statistical—manner. We will construct a functional  $\mathbb{F}[f_{ik}, F_{ij}, p_{ik}(\rho_i), P_{ij}(\rho_i, \Pi_j)]$  that will serve as a measure of the distance between the data and the probabilities. It should be maximal for states  $\rho_i$  which produce prior probabilities  $p_{ik}$  close to prior data  $f_{ik}$  as possible and simultaneously it should be maximal for and POVM  $\Pi_j$  which produces probabilities  $P_{ij}$  as close to the desired data  $F_{ij} = \delta_{ij}$  as possible. There are many convex and additive measures of distance between probability distributions. We will choose the one that provides the most simple extremal equations—the log-

likelihood functional,

$$\log \mathbb{L}[\rho_i, \Pi_j] = \sum_{ij} \delta_{ij} \ln P_{ij} + \sum_{ik} f_{ik} \ln p_{ik}. \quad (9)$$

Log likelihood as a measure of distance between quantum mechanical objects and real data arises from non-normalized multinomial distribution

$$\mathbb{L}[\rho_i, \Pi_j] \approx \prod_{ijk} P_{ij}^{\delta_{ij}} P_{ik}^{f_{ik}} = \prod_{ik} P_{ii} P_{ik}^{f_{ik}} \quad (10)$$

and has unique properties from the physical [26] as well as mathematical [27] point of view.

Variation of the log-likelihood functional (9),

$$\log \mathbb{L}[\rho_i + X_i, \Pi_j + Y_j] - \log \mathbb{L}[\rho_i, \Pi_j] = 0, \quad \forall X_i, Y_j, \quad (11)$$

that yields extremal equations for the states  $\rho_i$  and the optimal POVM  $\Pi_j$  must preserve trace normalization of the states,  $\text{Tr}[\rho_i] = 1$ , and completeness of the POVM,  $\sum_j \Pi_j = 1$ . These necessary constraints can be incorporated with the help of undetermined Lagrange multipliers. The functional to be maximized becomes

$$\mathbb{F}[\rho_i, \Pi_j] = \sum_{ij} \delta_{ij} \ln P_{ij} + \sum_{ik} f_{ik} \ln p_{ik} - \sum_i \mu_i \text{Tr}[\rho_i] - \text{Tr} \left[ \lambda \sum_j \Pi_j \right], \quad (12)$$

where  $\rho_i$  and  $\Pi_j$  are the independent variables. The estimated unknown states  $\rho_i^{\text{est}}$  and the designed optimal discrimination POVM  $\Pi_j^{\text{opt}}$  which maximize the functional (12) are solutions of the extremal equations

$$\mu_i^{-2} R_i \rho_i R_i = \rho_i, \quad \lambda^{-1} S_j \Pi_j S_j \lambda^{-1} = \Pi_j, \quad (13)$$

where

$$R_i = \frac{1}{P_{ii}} \Pi_i + \sum_k \frac{f_{ik}}{P_{ik}} \pi_k, \quad S_j = \frac{1}{P_{jj}} \rho_j, \quad (14)$$

$$\mu_i = (\text{Tr}[R_i \rho_i R_i])^{1/2}, \quad \lambda = \left( \sum_j S_j \Pi_j S_j \right)^{1/2}. \quad (15)$$

Notice that  $\lambda$  is positive definite operator and  $\mu_i$  are positive numbers. The set of nonlinear operator equations (13)–(15) can be solved by means of repeated

iterations,

$$\begin{aligned} (\mu_i^{(n)})^{-2} R_i^{(n)} \rho_i^{(n)} R_i^{(n)} &= \rho_i^{(n+1)}, \\ (\lambda^{(n)})^{-1} S_j^{(n)} \Pi_j^{(n)} S_j^{(n)} (\lambda^{(n)})^{-1} &= \Pi_j^{(n+1)}. \end{aligned} \quad (16)$$

The convexity of the log likelihood functional guarantees that these equations have only one global maximum or a plateau of global maxima. As initial iterations we can choose maximally mixed states and POVM elements  $\Pi_j^{(0)} = \frac{1}{J} 1$ . Let us note that when the true states  $\rho_i$  are mixed the estimated states  $\rho_i^{\text{est}}$  (13)–(15) are biased towards pure states due to the simultaneous construction of the discrimination POVM  $\Pi_j$ . The reason is that, roughly speaking, the rays are what is important for discrimination, and pure states can be discriminated easier than mixed ones. In another words, the proposed method finds the discrimination POVM  $\Pi_j^{\text{opt}}$  *optimal in the sense of maximum likelihood*, and simultaneously reconstructs rays corresponding to the unknown states.

#### 4. Examples

Here we illustrate our procedure on discrimination between two qubit states  $\rho_{1,2}$ , such as two spin states of an electron or two polarization states of a photon. We perform numerical simulations of the prior measurements and we find subsequently the states  $\rho_{1,2}^{\text{est}}$  and determined the optimal POVM  $\Pi_{1,2}^{\text{opt}}$  via iterative solution of the extremal equations (13)–(15). In our simulations we assume the class of states (5) and study the dependence of the error rate (2) on angle  $\alpha$  that controls the overlap between the two discriminated states  $\rho_{1,2}$ . The error rate (2) computed from the true states  $\rho_{1,2}$  and the optimal POVM  $\Pi_{1,2}^{\text{opt}}$  is used for evaluating the quality of the designed POVM  $\Pi_{1,2}^{\text{opt}}$ . Thus the optimality of the discrimination POVM  $\Pi_{1,2}^{\text{opt}}$  obtained with the help of the proposed method is quantified by the *standard* widely used measure of discrimination success.

In the first example we consider two symmetric mixed states (5) with  $d = 0.9$  on the Bob's side of the communication channel. The prior POVM  $\pi_k$  consists of projective measurements in  $x$  and  $y$  directions, each made on  $N = 1000$  physical systems in states  $\rho_{1,2}$ . This represents the calibration stage. Observation

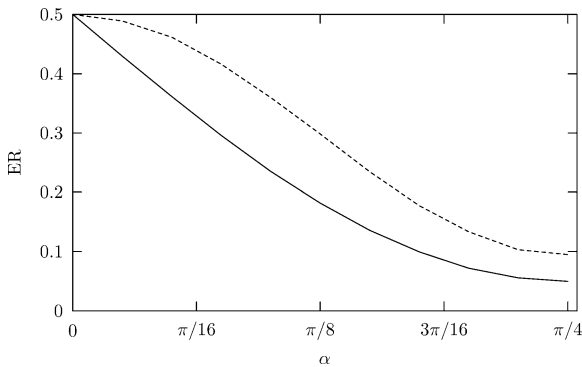


Fig. 2. Error rate of the optimal discrimination between two symmetric mixed states using partial prior information from the tomographically incomplete measurement (dashed), see text. Prior data  $f$  fluctuate around their theoretical values  $p$  with standard deviation  $p(1-p)/\sqrt{N}$ ,  $N = 1000$ . Error rate is limited by Helstrom bound (solid).

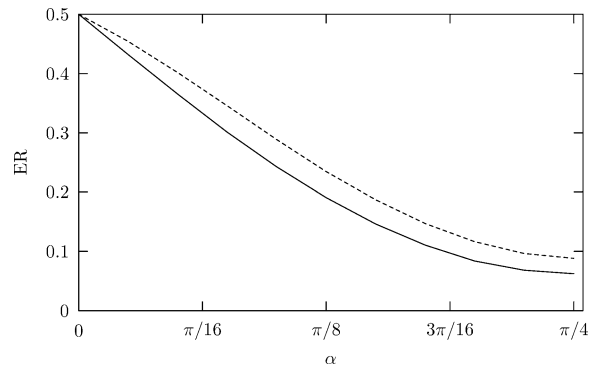


Fig. 3. The error rate of the optimal discrimination between a pure and a mixed state using the partial prior information from the tomographically complete measurement (dashed), see text. Prior data  $f$  fluctuate around their theoretical values  $p$  with standard deviation  $p(1-p)/\sqrt{N}$ ,  $N = 10$ . Error rate is limited by Helstrom bound (solid).

$\{\pi_x, \pi_{-x}, \pi_y, \pi_{-y}\}$  is tomographically incomplete and thus not sufficient for estimation of mixed states. Nevertheless, this measurement yields some information on the states  $\rho_{1,2}$  and, discrimination with the error rate less than 50% is possible. The error rate (2),  $\frac{1}{2}(\text{Tr}[\Pi_1^{\text{opt}} \rho_2] + \text{Tr}[\Pi_2^{\text{opt}} \rho_1])$ , of the optimal discrimination POVM  $\Pi_{1,2}^{\text{opt}}$  is shown in Fig. 2.

In the second example we discriminate between the pure state  $\rho_1 = |\psi_1\rangle\langle\psi_1|$ ,  $d = 1$ , and the mixed state  $\rho_2$  (5) with  $d = 3/4$ , having prior data from the tomographically complete observation  $\{\pi_x, \pi_{-x}, \pi_y, \pi_{-y}, \pi_z, \pi_{-z}\}$ . Each of these projective measurements in  $x$ ,  $y$  and  $z$  directions are done with only  $N = 10$  physical systems in states  $\rho_1$  and  $\rho_2$ . The optimal POVM  $\Pi_{1,2}^{\text{opt}}$  can be found with the help of (16). The corresponding error rate is shown in Fig. 3. In the case of an infinitely large number of physical systems used for the prior measurement in the calibration stage the prior data  $f_{ik}$  attain the theoretical probabilities  $p_{ik}$  and the error rate approaches the bound,  $\text{ER} = \frac{1}{2}(1 - \frac{1+d}{2} \sin 2\alpha) = \frac{1}{2}(1 - \frac{7}{8} \sin 2\alpha)$ , as is shown in Fig. 3 by solid line.

### 5. Conclusion

Simultaneous reconstruction of quantum states and design of optimal quantum discrimination POVM have been presented. Observation on finite samples of physical systems in unknown quantum states in

the calibration stage provides the only prior information used for discrimination between these quantum states. The system of the extremal equations (13)–(15) fully determines the unknown states and the optimal POVM for their discrimination. These nonlinear operator equations can be solved by means of repeated iterations. This has been demonstrated on particular examples and verified in many other cases. Typically, several hundreds or thousands of iterations are needed for  $10^{-16}$  accuracy. In the case of finite samples of physical systems used for the calibration the Helstrom bound represents a lower limit of the achievable error rate. For tomographically complete prior observation the error rate can come very close to this bound even if the prior measurements are carried out on a small number of states. This means that in practice Alice and Bob need only a few transmitted quantum states for establishing nearly optimal discrimination device. The calibration stage thus can be very short and almost all the states reserved for communication can be used up for transmission of useful data from Alice to Bob. Of course the ratio of the transmitted calibration states to the states used for communication can be adjusted in order to achieve either transmission with maximal speed for the given error rate or transmission with minimal error rate for the given speed. These and others optimization problems can be formulated as a modifications of the presented quantum discrimination protocol.

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