

Optimal discrimination of mixed quantum states involving inconclusive results

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We propose a generalized discrimination scheme for mixed quantum states. In the present scenario we allow for certain fixed fraction of inconclusive results and we maximize the success rate of the quantum-state discrimination. This protocol interpolates between the Ivanovic-Dieks-Peres scheme and the Helstrom one. We formulate the extremal equations for the optimal positive operator valued measure describing the discrimination device and establish a criterion for its optimality. We also devise a numerical method for efficient solving of these extremal equations.

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I. INTRODUCTION

The nonorthogonality of quantum states is one of the fundamental features of quantum mechanics. It means that two different quantum states cannot be perfectly discriminated in general. This fact imposes limits on the processing of quantum information. For instance, an unknown quantum state cannot be copied [1,2]. On the other hand, the nonorthogonality of the quantum states can be advantageous. In particular, most of the quantum cryptographic schemes rely on it [3–5]. The principle of operation of these schemes is to employ several nonorthogonal states to encode the bits of the secret key that is sent from transmitter (Alice) to receiver (Bob). Since the eavesdropper (Eve) cannot discriminate all these states perfectly, any attempt to gain information on the key will increase the noise of the transmission from Alice to Bob and thus Eve will be unavoidably revealed.

Since one cannot discriminate nonorthogonal states perfectly, a natural question arises what is the optimal approximate discrimination scheme. Two different approaches to the problem have been proposed. The first scenario, the ambiguous quantum state discrimination, has been analyzed by Holevo, Helstrom, and others [6–13]. They considered discrimination of N mixed quantum states ρ_j that are generated by Alice with the *a priori* probabilities p_j ,

$$\sum_{j=1}^N p_j = 1, \quad p_j > 0. \quad (1)$$

Bob performs a generalized quantum measurement described by the N -component positive operator valued measure (POVM) $\{\Pi_j\}_{j=1}^N$. If the outcome Π_j is detected, then Bob concludes that Alice sent the state ρ_j to him. The optimal POVM is defined as the POVM that maximizes the average success rate of the Bob's guesses

$$P_S = \sum_{j=1}^N p_j \text{Tr}[\Pi_j \rho_j]. \quad (2)$$

An interesting alternative approach to the discrimination of two pure states was suggested by Ivanovic, Dieks, and Peres (IDP) [14–17]. They showed that these states can be discriminated unambiguously, provided that we allow for some

fraction of the inconclusive results P_I . Recently, the IDP scenario was extended to N linearly independent pure states by several authors [18–22]. Note, that several other problems closely related to quantum-state discrimination have been considered such as the discrimination of sets of quantum states [23,24] and the quantum-state comparison [25,26].

Besides being a theoretically interesting problem, the quantum-state discrimination has also found practical applications in quantum information processing. In particular, the optimal unambiguous quantum-state discrimination represents a simple and experimentally feasible attack on the quantum key distribution protocols [27,28]. Recently, some optimal discrimination POVMs have been realized experimentally and the Helstrom as well as the IDP bounds have been attained [29–32].

The two above discussed scenarios can be considered as limiting cases of a more general scheme that involves certain fraction of inconclusive results P_I for which we maximize the success rate. The main feature of this scheme is that if we allow for inconclusive results then we can improve the relative (or renormalized) success rate

$$P_{RS} = \frac{P_S}{1 - P_I}. \quad (3)$$

In other words, with probability P_I Bob fails completely and he cannot say at all which state was sent to him. However, in the rest of the cases when he succeeds he can correctly guess the state with higher probability than if he would not allow for the inconclusive results. For pure linearly independent states this generalized scenario was discussed in two recent papers [33,34].

Here we extend the analysis to general mixed states. At a first sight, such an extension may seem to be a bit problematic, because it is known that one cannot unambiguously discriminate mixed states (the reason is that the IDP scheme does not work for linearly dependent states). Nevertheless, we shall show that the extension is perfectly meaningful also for mixed states. Although we cannot reach the limit of perfect unambiguous discrimination, we can improve the P_{RS} and we shall derive an upper bound on P_{RS} that can be achieved for a given set of mixed states.

II. EXTREMAL EQUATIONS FOR OPTIMAL POVM

Let us begin with the formal definition of the problem. We assume that the quantum state sent to Bob is drawn from the set of N mixed states $\{\rho_j\}_{j=1}^N$ with the *a priori* probabilities p_j . Bob's measurement on the state may yield $N+1$ different results and is formally described by the POVM whose $N+1$ components satisfy $\Pi_j \geq 0, j=0, \dots, N$, and

$$\sum_{j=0}^N \Pi_j = \mathbb{1}, \tag{4}$$

where $\mathbb{1}$ is the identity operator. The outcome Π_0 indicates failure and the probability of inconclusive results is thus given by

$$P_I = \sum_{j=1}^N p_j \text{Tr}[\rho_j \Pi_0]. \tag{5}$$

For a certain fixed value of P_I we want to maximize the relative success rate (3) which is equivalent to the maximization of the success rate (2). To account for the linear constraints (4) and (5) we introduce Lagrange multipliers λ and a , where λ is Hermitian operator and a is a real number. Taking everything together we should maximize the constrained success rate functional

$$\bar{P}_S = \sum_{j=1}^N p_j \text{Tr}[\rho_j \Pi_j] - \sum_{j=0}^N \text{Tr}[\lambda \Pi_j] + a \sum_{j=1}^N p_j \text{Tr}[\rho_j \Pi_0]. \tag{6}$$

We now derive the extremal equations that must be satisfied by the optimal POVM. We expand the POVM elements in terms of their eigenstates and eigenvalues, $\Pi_j = \sum_k r_{jk} |\varphi_{jk}\rangle\langle\varphi_{jk}|$ and vary Eq. (6) with respect to $\langle\varphi_{jk}|$. After some algebraic manipulations, we arrive at the extremal equations,

$$(\lambda - p_j \rho_j) \Pi_j = 0, \quad j=1, \dots, N, \tag{7}$$

$$(\lambda - a\sigma) \Pi_0 = 0, \tag{8}$$

where the operator σ introduced for the sake of notational simplicity reads

$$\sigma = \sum_{j=1}^N p_j \rho_j. \tag{9}$$

From the constraint $\text{Tr}[\sigma \Pi_0] = P_I$ we can express a in terms of λ ,

$$a = P_I^{-1} \text{Tr}[\lambda \Pi_0]. \tag{10}$$

Furthermore, if we sum all Eq. (7) and also Eq. (8) and use the resolution of the identity (4), we obtain formula for λ ,

$$\lambda = \sum_{j=1}^N p_j \rho_j \Pi_j + a \sigma \Pi_0. \tag{11}$$

If we combine Eqs. (10) and (11) then we can express a and λ solely in terms of p_j, ρ_j , and Π_j . This may be important, for example, if we guess the optimal POVM and want to determine the corresponding Lagrange multipliers. The extremal Eqs. (7) and (8) constitute a generalization of the extremal equations for optimal POVM for ambiguous quantum-state discrimination that were derived by Holevo and Helstrom [6,8].

We now provide simple sufficient conditions on the optimality of the POVM. If the POVM satisfies the extremal Eqs. (7) and (8) and if the following inequalities hold

$$\lambda - p_j \rho_j \geq 0, \quad j=1, \dots, N, \tag{12}$$

$$\lambda - a\sigma \geq 0, \tag{13}$$

then the POVM is the optimal one that maximizes the success rate P_S for a given fixed probability of inconclusive results P_I .

To prove this statement we show that the Lagrange multipliers provide an upper bound on the success rate and that this bound is saturated by the POVM that satisfies Eqs. (7) and (8). From the definition of the success rate (2), the inequalities (12) and the normalization (4) we obtain

$$P_S \leq \sum_{j=1}^N \text{Tr}[\lambda \Pi_j] = \text{Tr}[\lambda(1 - \Pi_0)]. \tag{14}$$

Now we use the inequality (13) and finally we take into account the constraint $\text{Tr}[\sigma \Pi_0] = P_I$. We arrive at

$$P_S \leq \text{Tr}[\lambda] - a P_I. \tag{15}$$

This last inequality shows that P_S is limited from above by the quantity that depends only on the Lagrange multipliers λ and a and also on the fixed P_I . If the POVM Π_j satisfies the extremal Eqs. (7) and (8) then this upper bound is reached, as can easily be checked.

We have thus established a simple criterion for checking of the POVM optimality. Of course, we would like to derive the optimal POVM Π_j for given p_j, ρ_j , and P_I . The analytical solution to this problem seems to be extremely complicated. Nevertheless, recently it was pointed out that one can solve this kind of problems very efficiently numerically [35]. One possible simple and fruitful approach is to solve the extremal equations by means of repeated iterations [35–40]. In principle, one could iterate directly Eqs. (7) and (8). However, the POVM elements Π_j should be positive semidefinite Hermitian operators. All constraints can be exactly satisfied at each iteration step if the extremal equations are symmetrized. First we express $\Pi_j = p_j \lambda^{-1} \rho_j \Pi_j$ and combine it with its Hermitian conjugate. We proceed similarly also for Π_0 and we get

$$\Pi_j = p_j^2 \lambda^{-1} \rho_j \Pi_j \rho_j \lambda^{-1}, \quad j=1, \dots, N, \tag{16}$$

$$\Pi_0 = a^2 \lambda^{-1} \sigma \Pi_0 \sigma \lambda^{-1}. \tag{17}$$

The Lagrange multipliers λ and a must be determined self-consistently so that all the constraints will hold. If we sum Eqs. (16) and (17) and take into account that $\sum_{j=0}^N \Pi_j = 1$, we obtain

$$\lambda = \left[\sum_{j=1}^N p_j^2 \rho_j \Pi_j \rho_j + a^2 \sigma \Pi_0 \sigma \right]^{1/2}. \quad (18)$$

The fraction of inconclusive results calculated for the POVM after the iteration is given by

$$P_I = a^2 \text{Tr}[\sigma \lambda^{-1} \sigma \Pi_0 \sigma \lambda^{-1}]. \quad (19)$$

Since the Lagrange multiplier λ is expressed in terms of a , Eq. (19) forms a nonlinear equation for a single real parameter a (or, more precisely, a^2). This nonlinear equation can be very efficiently solved by Newton's method of halving the interval. At each iteration step for the POVM elements, we thus solve the system of coupled nonlinear Eqs. (18) and (19) for the Lagrange multipliers. These self-consistent iterations work very well and our extensive numerical calculations confirm that they typically exhibit an exponentially fast convergence [35].

We note that the maximization of the success rate P_S for a fixed fraction of inconclusive results P_I can also be formulated as a semidefinite program. Powerful numerical methods developed for solving this kind of problems may be applied. Here we will not investigate this issue in detail and we refer the reader to the papers [35,41,42] where the formulation of optimal quantum-state discrimination as a semidefinite program is described in detail. Note also that the semidefinite programming has recently found its applications in several branches of quantum information theory such as the optimization of completely positive maps [43–45], the analysis of the distillation protocols that preserve the positive partial transposition [46], or the tests of separability of quantum states [47].

III. MAXIMAL ACHIEVABLE RELATIVE SUCCESS RATE

As the fraction of inconclusive results is increased the success rate P_S decreases. However, the relative success rate P_{RS} grows until it achieves its maximum. If $\{\rho_j\}_{j=1}^N$ are linearly independent pure states, then $P_{RS,\max} = 1$ because exact IDP scheme works and the unambiguous discrimination is possible. Generally, however the maximum is lower than unity. To find this maximum, we notice that in the limit $P_I \rightarrow 1$ the POVM element Π_0 must tend to the identity operator. This means that at some point Π_0 becomes positive definite operator and all its eigenvalues are strictly positive. In that case, the extremal Eq. (8) can be satisfied if and only if

$$\lambda = a\sigma. \quad (20)$$

Since we are looking for some nontrivial solution to the extremal equations with $P_I < 1$, at least one of the extremal Eqs. (7) must have a nontrivial solution $\Pi_j \neq 0$. This implies that at least one of the operators $\lambda - p_j \rho_j$ must have one eigenvalue μ equal to zero which implies that

$$\det[a\sigma - p_j \rho_j] = 0 \quad (21)$$

must hold at least for one of the states ρ_j . The optimal Π_j is then proportional to the projector to the subspace spanned by eigenvectors corresponding to the eigenvalue $\mu = 0$.

The maximal attainable relative success rate is obtained if we insert Eq. (20) into Eq. (15), taken into account that $\text{Tr}[\sigma] = \sum_j p_j = 1$ and renormalize according to Eq. (3),

$$P_{RS} = a. \quad (22)$$

To determine the maximal P_{RS} we must find the maximal a that satisfies Eq. (21). Since σ is positive definite it can be inverted, and we can equivalently express the maximal a as the maximal eigenvalue of a Hermitian matrix,

$$a_j = p_j \max[\text{eig}(\sigma^{-1/2} \rho_j \sigma^{-1/2})]. \quad (23)$$

The maximal P_{RS} is equal to the largest a_j ,

$$P_{RS,\max} = \max_j a_j. \quad (24)$$

For qubits, Eq. (21) leads to quadratic equation for the multiplier a_j that can be solved analytically,

$$(a_j - p_j)^2 = a_j^2 \text{Tr}[\sigma^2] - 2a_j p_j \text{Tr}[\sigma \rho_j] + p_j^2 \text{Tr}[\rho_j^2]. \quad (25)$$

It turns out that the maximal P_{RS} depends only on the *a priori* probabilities p_j , the purities of the states ρ_j ,

$$\mathcal{P}_j = \text{Tr}[\rho_j^2] \quad (26)$$

and the overlaps of states ρ_j and ρ_k ,

$$\mathcal{O}_{jk} = \text{Tr}[\rho_j \rho_k]. \quad (27)$$

In this context it is worth noting that it was shown recently that these parameters of the quantum states can directly be measured without the necessity to carry out a complete quantum state reconstruction [48–50].

IV. DISCRIMINATION BETWEEN TWO MIXED QUBIT STATES

We proceed to illustrate the methods developed in the present paper on explicit example. We consider the simple yet nontrivial problem of optimal discrimination between two mixed qubit states ρ_1 and ρ_2 . To simplify the discussion, we shall assume that the purities of these states as well as the *a priori* probabilities are equal, $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$, $p_1 = p_2 = 1/2$. The mixed states can be visualized as points inside the Poincaré sphere and the purity determines the distance of the point from the center of that sphere. Without loss of generality, we can assume that both states lie in the xz plane and are symmetrically located about the z axis,

$$\rho_{1,2} = \eta \psi_{1,2}(\theta) + \frac{1-\eta}{2} 1, \quad (28)$$

where $\psi_j = |\psi_j\rangle\langle\psi_j|$ denotes a density matrix of a pure state,

$$|\psi_{1,2}(\theta)\rangle = \cos \frac{\theta}{2} |0\rangle \pm \sin \frac{\theta}{2} |1\rangle, \quad (29)$$

and $\theta \in (0, \pi/2)$. The parameter η determines the purity of the mixed state (28), $\mathcal{P} = (1 + \eta^2)/2$.

From the symmetry it follows that the elements Π_1 and Π_2 of the optimal POVM must be proportional to the projectors $\psi_1(\phi)$ and $\psi_2(\phi)$, where the angle $\phi \in (\pi/2, \pi)$ is related to the fraction of the inconclusive results. The third component Π_0 is proportional to the projector onto state $|0\rangle$. The normalization of the POVM elements can be determined from the constraint (4) and we find

$$\Pi_{1,2}(\phi) = \frac{1}{2 \sin^2(\phi/2)} \psi_{1,2}(\phi), \quad (30)$$

$$\Pi_0(\phi) = \left(1 - \frac{1}{\tan^2(\phi/2)}\right) |0\rangle\langle 0|.$$

The relative success rate for this POVM reads

$$P_{RS} = \frac{1 + \eta \cos(\phi - \theta)}{2(1 + \eta \cos \theta \cos \phi)} \quad (31)$$

and the fraction of inconclusive results is given by

$$P_I = \frac{1}{2} (1 + \eta \cos \theta) \left(1 - \frac{1}{\tan^2(\phi/2)}\right). \quad (32)$$

The formulas (31) and (32) describe implicitly the dependence of the relative success rate P_{RS} on the fraction of the inconclusive results P_I . From Eqs. (10) and (11) one can determine the Lagrange multipliers λ and a for the POVM (30) and check that the extremal Eqs. (7), (8), (12), and (13) are satisfied which proves that the POVM (30) is indeed optimal.

The maximum $P_{RS, \max}$ (24) is achieved if the angle ϕ is chosen as follows:

$$\cos \phi_{\max} = -\eta \cos \theta. \quad (33)$$

On inserting the optimal ϕ_{\max} back into Eq. (31) we get

$$P_{RS, \max} = \frac{1}{2} \left[1 + \frac{\eta \sin \theta}{\sqrt{1 - \eta^2 \cos^2 \theta}} \right]. \quad (34)$$

Making use of Eqs. (24) and (25) we can express the $P_{RS, \max}$ in terms of the overlap \mathcal{O}_{12} and the purity \mathcal{P} ,

$$P_{RS, \max} = \frac{1}{2} \left[1 + \sqrt{\frac{\mathcal{P} - \mathcal{O}_{12}}{2 - \mathcal{P} - \mathcal{O}_{12}}} \right]. \quad (35)$$

If we calculate \mathcal{O}_{12} and \mathcal{P} for the density matrices (28) and insert them into Eq. (35) then we recover the formula (34). The optimal POVM (30) can be also obtained numerically. We demonstrate the feasibility of iterative solution of the

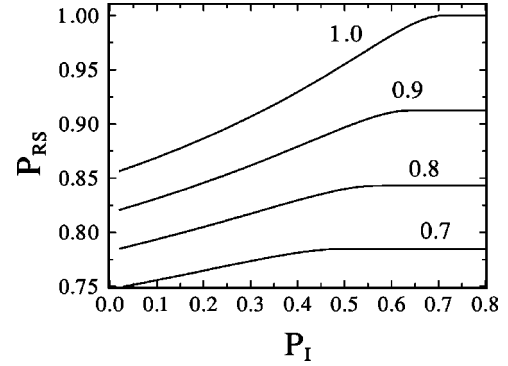


FIG. 1. Relative success rate P_{RS} versus the fraction of inconclusive results P_I for the optimal discrimination of two mixed states (28) with $\theta = \pi/4$ and four different parameters $\eta = 0.7$, $\eta = 0.8$, $\eta = 0.9$, and $\eta = 1.0$.

symmetrized extremal Eqs. (16), (17), (18), and (19) for mixed quantum states (28) with the angle of separation $\theta = \pi/4$. The trade-off of the relative success rate and the probability of inconclusive results is shown in Fig. 1 for various purities of the states being discriminated. For the given probability P_I of inconclusive results and the given purity of the states the extremal equations are solved self-consistently by means of repeated iterations. The success rate P_S is calculated from the obtained optimal POVM and renormalized according to Eq. (3). The numerically obtained dependence of P_{RS} on P_I is in excellent agreement with the analytical dependence following from formulas (31) and (32). Typically, a sixteen digit precision is reached after several tens of iterations. The trade-off curves shown in Fig. 1 reveal the monotonic growth of P_{RS} until the maximal plateau (34) is reached.

V. CONCLUSIONS

In conclusion, we have considered a generalized discrimination scheme for mixed quantum states. The present scenario interpolates between the Helstrom and IDP schemes. We allow for certain fixed fraction of inconclusive results and maximize the success rate. We have derived the extremal equations for the optimal POVM that describes the discrimination device. The extremal equations can be very efficiently solved numerically by means of the devised simple iterative algorithm or, alternatively, by using the powerful techniques of semidefinite programming.

We have showed that the relative success rate P_{RS} monotonically grows as the fraction of inconclusive results P_I is increased and at certain point it reaches its upper bound $P_{RS, \max}$. For pure linearly independent states this bound is $P_{RS, \max} = 1$ which corresponds to the IDP unambiguous discrimination scheme. For mixed states this bound is in general lower than unity and we have derived a simple formula for it.

The present scheme may be important for quantum cryptographic schemes where the receiver and(or) eavesdropper want to discriminate nonorthogonal states. Although these schemes are usually based on pure states, in realistic cases the unavoidable noise and decoherence will reduce the purity

of these states and one would have to deal with mixed states. Various modifications of our method can be suggested for such application. For instance, if the involved states are in some sense asymmetric, one may impose the condition that the probabilities of inconclusive results or successful discrimination of the states ρ_j should all be identical, and minimize the error rate with this additional constraint.

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