How do two moving clocks fall out of sync? A tale of trucks, threads, and twins

Daniel F. Styer[a)]
Department of Physics and Astronomy, Oberlin College, Oberlin, Ohio 44074

(Received 18 August 2006; accepted 30 March 2007)

In special relativity, a pair of clocks synchronized in their own reference frame are not synchronized in another. How do two clocks, initially synchronized and at rest in the laboratory frame, fall out of sync as their speed relative to the lab gradually increases? The answer lies in general-relativistic time dilation. The path to the answer sheds light on the thread-between-spaceships paradox (also called the Bell spaceship paradox), on the twin paradox, and on the character of length contraction.
[DOI: 10.1119/1.2733691]

I. INTRODUCTION AND QUALITATIVE OVERVIEW

The statement that "two events simultaneous in one reference frame might not be simultaneous in another" is the central claim of special relativity. At the same time, it is the most difficult claim to accept psychologically.[1,2]

Phrased quantitatively, the claim is: In one frame two events are simultaneous and separated by distance Δx. In a frame moving relative to the first at velocity V, the two events are separated in time by

$$\frac{-V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \tag{1}$$

(All motion and events are restricted to the x axis. We use V for the speed of a frame and v for the speed of a clock.) A logically equivalent statement is: Two clocks, synchronized and separated by distance L_0 in their rest frame, are not synchronized in a frame where the two clocks move at velocity V. Instead, the rear clock is set ahead by $L_0 V/c^2$.

The second form of the claim has several advantages. While the first form encourages blind plug-and-chug, the second encourages thoughtful questions such as "How do those two moving clocks fall out of sync?" (This question is very different from the one answered by the synchronization principle: The synchronization principle compares a pair of clocks in their own frame with that same pair in a frame moving relative to those clocks, while the question compares a pair of clocks in a single frame before and after acceleration.) After all, if two clocks start out synchronized and at rest in the laboratory, and if they execute identical acceleration programs (identical in the laboratory frame), then throughout the acceleration process the two moving clocks tick slowly (relative to laboratory clocks). *But each clock ticks slowly by the same factor* so the two clocks remain synchronized in the laboratory frame [see Fig. 1(a)].

The resolution lies in general-relativistic time dilation.[3] Consider this acceleration process in the (non-inertial) frame of the right-hand clock [see Fig. 1(b)]. During the acceleration, the two clocks behave (principle of equivalence) as if they are in a gravitational field (with "down" being to the left in the figure). Therefore (gravitational time dilation) the left-hand ("lower") clock ticks slower than the right-hand ("upper") clock. At the end of the acceleration process, the two clocks are *not* synchronized in their own frame: the left-hand clock has ticked off less time than the right-hand clock has.

So, after acceleration, the two clocks are in sync in the laboratory frame, but not in their own frame. The situation doesn't conform to the antecedent "Two clocks, synchronized … in their own rest frame" of the statement directly following Eq. (1). To make them conform, the "master time keeper" of the clock's frame (in charge of keeping clocks in sync) must manually set forward the time reading on the left-hand clock. This last step, which synchronizes the two clocks in their own frame, pushes the clocks out of sync in the laboratory frame, with the rear clock set ahead.

We gain insight by examining not only the clock readings but also the distance between clocks. In the laboratory frame each clock follows an identical acceleration program and thus, of mathematical necessity, the distance between the two clocks remains constant. In the frame of the right-hand clock [see Fig. 1(b)], the left-hand clock moves left so the two clocks draw apart. How can this be? In the laboratory frame, the two clocks simultaneously reach the state of, for example, "clock reading $\tau=13.1$ s, speed $v=0.718c$" [this instant is depicted in Fig. 1(a) as "during"]. In the inertial frame moving relative to the lab at $V=0.718c$, these two events are *not* simultaneous: First the right-hand clock reaches the state of "clock reading $\tau=13.1$ s, speed $v=0$" [this instant is depicted in Fig. 1(b) as "during"], and then some time later the left-hand clock reaches that state. At the frame and instant depicted in Fig. 1(b), the left-hand clock has not yet reached the reading of $\tau=13.1$ s and not yet achieved a speed of $v=0$, so it is still moving toward the left.

Details and formulas will be derived in Sec. II, but already we have raised a puzzle about length contraction. Standard treatments[4] of length contraction compare the length of a truck in its own frame with the length of that truck in a frame moving relative to that truck, correctly showing that the truck is shorter in the second frame.

But standard treatments have nothing to say about the length of a truck before and after acceleration. Often it is assumed that the truck maintains its same proper length through the acceleration process and hence shrinks in the lab frame.[5] But the preceding analysis shows that if both the nose and the tail of a truck undergo identical acceleration programs, then the truck maintains the same length in the laboratory frame while its proper length increases. [Both possibilities are consistent with the standard length contraction conclusion that the truck is shorter in a frame moving relative to the truck (in this case the lab frame) than it is in its own proper frame.] To answer the question of the truck's

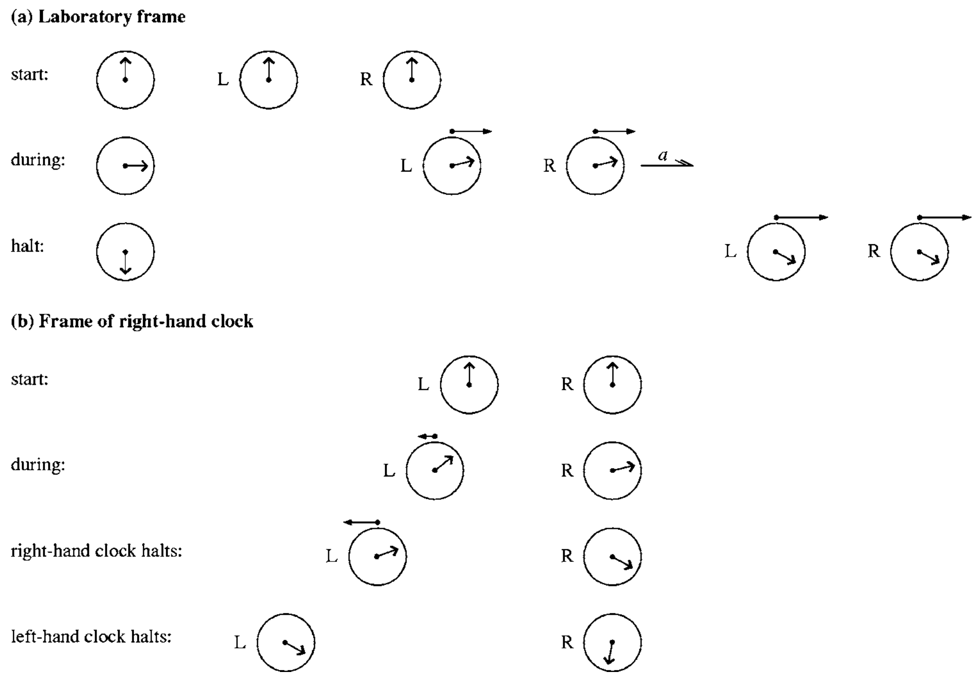

Fig. 1. (a) In the laboratory frame, two clocks accelerate (with constant proper acceleration) from $v=0$ to $v=0.9c$ in 30 s. Throughout the process the two clocks are synchronized and separated by distance (5 s)c. (b) Same process observed in three inertial frames moving at the speed of the right-hand clock. [Characteristically, one picture in the lab frame ("both clocks halt acceleration") corresponds to two pictures in the frame moving at $V=0.9c$ relative to the lab (first "right-hand clock halts acceleration" and some time later "left-hand clock halts acceleration").] At the end of the process, the clocks are not synchronized and they are separated by a longer distance $(5\text{ s})c/\sqrt{1-(0.9)^2}$.

proper length after acceleration, we must know not only the final velocity of the truck, but also some details about how the acceleration is carried out.

Sections II and III do little more than produce the equations needed to generate and confirm Fig. 1. Section IV amplifies the remarks about length contraction, and Secs. V and VI apply these ideas to the thread and twin paradoxes. Throughout this analysis we assume, as has been verified by experiment,[6] that an accelerated clock ticks at precisely the same rate as an instantaneously co-moving non-accelerating clock. Although no problem discussed in this paper *requires* the use of general relativity, this additional perspective provides both insight and satisfaction.

II. ACCELERATION PROCESS: SPECIAL RELATIVISTIC APPROACH

A. A single accelerating clock

A clock has initial position x_i and initial velocity v_i in the laboratory frame. It moves with constant acceleration in its own frame. How does it move in the laboratory frame?

If a clock has velocity v and acceleration a in the lab frame, then in a frame moving relative to the lab at velocity V it has acceleration[7,8]

$$a' = \left[\frac{\sqrt{1-(V/c)^2}}{1 - vV/c^2} \right]^3 a. \tag{2}$$

If the velocity of the frame is the same as the velocity of the clock, then

$$a' = \frac{a}{[\sqrt{1-(v/c)^2}]^3}, \tag{3}$$

so if a' is constant (call it g, the constant proper acceleration), then in the lab frame

$$a = \frac{dv}{dt} = g[\sqrt{1-(v/c)^2}]^3. \tag{4}$$

Integrate this once to find, for initial velocity v_i,

$$v(t) = \frac{g(t-t_0)}{\sqrt{1+(g(t-t_0)/c)^2}}, \tag{5}$$

where

$$t_0 \equiv \frac{-v_i/g}{\sqrt{1-(v_i/c)^2}}. \tag{6}$$

Integrate again to find, for initial position x_i,

$$x(t) = \frac{c^2}{g}\left[\sqrt{1+\left(\frac{g(t-t_0)}{c}\right)^2} - 1 \right] + x_0, \tag{7}$$

where

$$x_0 \equiv \frac{c^2}{g}\left[1 - \frac{1}{\sqrt{1-(v_i/c)^2}} + \frac{x_i g}{c^2} \right]. \tag{8}$$

The worldline $x(t)$ of such a clock, for the case $v_i=0$, $x_i=0$, is sketched in Fig. 2.

The proper time τ ticked off by the clock after its motion starts is determined through

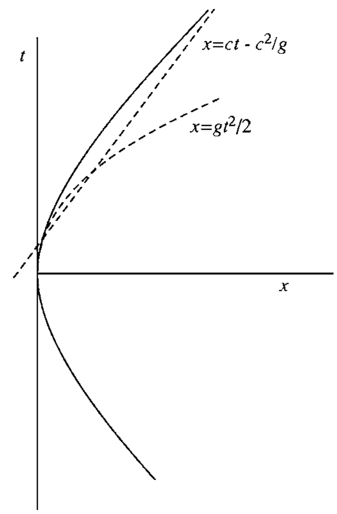

Fig. 2. Worldline of a clock undergoing constant proper acceleration g.

$$d\tau = \sqrt{1-(v/c)^2}dt = \sqrt{1-\left(\frac{g(t-t_0)/c}{\sqrt{1+(g(t-t_0)/c)^2}}\right)^2}dt, \quad (9)$$

$$\tau = \frac{c}{g}\left[\operatorname{arcsinh}\left(\frac{g(t-t_0)}{c}\right)+\operatorname{arcsinh}\left(\frac{gt_0}{c}\right)\right]. \quad (10)$$

How does this motion happen in reference frame S', which moves relative to the lab at velocity V? The motion has constant proper acceleration g in this frame, too, so the motion has the same *form* as Eqs. (5) and (7), but with different coordinates for the event of stillness (x_0, t_0). To find these coordinates we first find the event at which the clock has velocity V in the lab frame. It is

$$(\bar{x},\bar{t}) = \left(x_0 + \frac{c^2}{g}\left[\frac{1}{\sqrt{1-(V/c)^2}}-1\right], t_0 + \frac{V/g}{\sqrt{1-(V/c)^2}}\right). \quad (11)$$

Using the Lorentz transformation to find the coordinates of this event in frame S', we obtain

$$(\bar{x}',\bar{t}') = \frac{1}{\sqrt{1-(V/c)^2}}(x_0 - Vt_0 - (c^2/g)[1-\sqrt{1-(V/c)^2}],\ t_0 - Vx_0/c^2 + V/g). \quad (12)$$

In a slight abuse of notation we call the coordinates of this stillness event (x_0', t_0'), so the clock's trajectory in S' is

$$x'(t') = \frac{c^2}{g}\left[\sqrt{1+\left(\frac{g(t'-t_0')}{c}\right)^2}-1\right]+x_0'. \quad (13)$$

Finally, suppose that the motion is not constant proper acceleration for all time, but instead zero acceleration previous to lab time $t=0$, followed by constant proper acceleration g, followed by zero acceleration after lab time $t=t_f$. In this situation the coordinates of the initial and final acceleration events, and the initial and final velocities, are easily found in the lab frame and easily transformed into frame S'.

B. A pair of accelerating clocks

In the lab frame, two clocks start at rest at $t=0$ and move with constant proper acceleration g until $t=t_f$. The left-hand clock has initial position 0 while the right-hand clock has initial position ℓ_i. Hence the two clocks move in the laboratory frame according to

$$x_L(t) = \frac{c^2}{g}\left[\sqrt{1+\left(\frac{gt}{c}\right)^2}-1\right], \quad (14a)$$

$$x_R(t) = \ell_i + \frac{c^2}{g}\left[\sqrt{1+\left(\frac{gt}{c}\right)^2}-1\right] \quad (14b)$$

for $0<t<t_f$. They initially have speed 0, then accelerate to $gt/\sqrt{1+(gt/c)^2}$, and end with speed

$$v_f = \frac{gt_f}{\sqrt{1+(gt_f/c)^2}}. \quad (15)$$

How do these two clocks move in inertial frame S', which moves relative to the lab with velocity V $(0\le V\le v_f)$? Each clock initially has velocity $-V$, then slows to zero speed, then moves to the right, and finally halts acceleration with velocity

$$\frac{v_f - V}{1-v_f V/c^2}. \quad (16)$$

But in frame S' the two clocks don't start accelerating simultaneously: the right-hand clock starts accelerating first. Similarly, the right-hand clock has zero speed first, and the right-hand clock halts its acceleration first. Because $x_R(t)=\ell_i+x_L(t)$, the Lorentz transformation shows that the worldline of the right-hand clock on a spacetime diagram is obtained from the worldline of the left-hand clock by shifting it *right* by $\ell_i/\sqrt{1-(V/c)^2}$ and *down* by $(V\ell_i/c^2)/\sqrt{1-(V/c)^2}$. In other words (see Fig. 3),

$$x_R'(t') = x_L'(t' + (V\ell_i/c^2)/\sqrt{1-(V/c)^2}) + \ell_i/\sqrt{1-(V/c)^2}. \quad (17)$$

When the lab time is t each clock has ticked off proper time τ, where

$$\tau = \frac{c}{g}\operatorname{arcsinh}\left(\frac{gt}{c}\right) \quad (18)$$

or

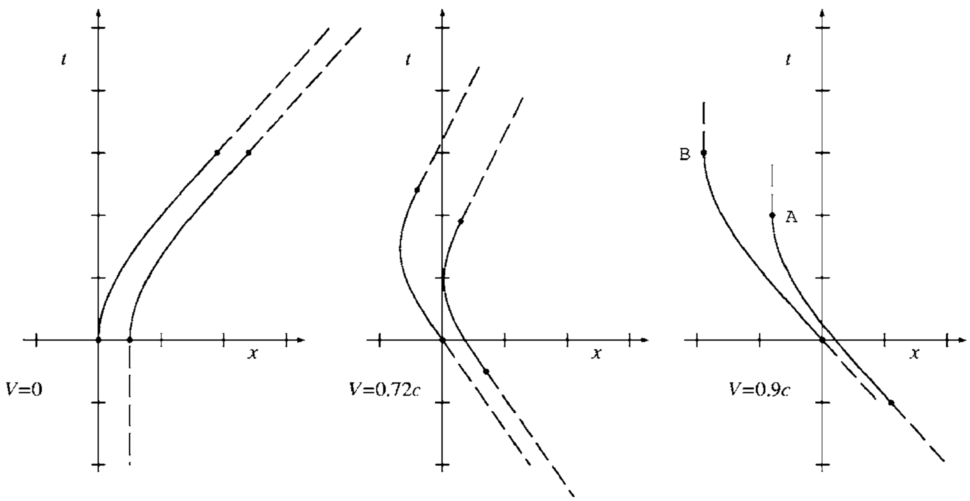

Fig. 3. In the laboratory frame, two clocks accelerate (with constant proper acceleration) from $v=0$ to $v=0.9c$ in 30 s. Each t hash mark corresponds to 10 s, each x hash mark to 10 light-seconds. Worldlines are shown in the lab frame and in two inertial frames moving relative to the lab frame at the constant velocities indicated.

$$\frac{gt}{c}=\sinh\left(\frac{g\tau}{c}\right). \tag{19}$$

Thus, in the lab frame, the clock readings τ are less than the lab time t, but at a given lab time both clocks have the same reading. This is *not* true in any other frame. It is true that in any frame, each clock will have the same reading when each reaches a specified velocity. But the right-hand clock will attain that velocity (and that reading) before the left-hand clock does. The situation "two clocks undergo identical acceleration programs in the lab frame" is the same as "two clocks undergo identical acceleration programs as determined by their own clocks." But such acceleration programs are not identical in any other frame.

There is no such thing as an "instantaneously co-moving inertial frame of both clocks." In the co-moving frame of the right-hand clock, the left-hand clock moves toward the left. In the co-moving frame of the left-hand clock, the right-hand clock moves toward the right.

In the frame[9] with $V=v_f$, each clock halts its acceleration at the same time that it reaches speed zero. The right-hand clock does so first, at the event with coordinates [see Eq. (12)]

$$(x'_A,t'_A)=\frac{1}{\sqrt{1-(V/c)^2}}(\ell_i-(c^2/g)[1-\sqrt{1-(V/c)^2}], \\ -V\ell_i/c^2+V/g). \tag{20}$$

The left-hand clock halts its acceleration later at the event with coordinates

$$(x'_B,t'_B)=\frac{1}{\sqrt{1-(V/c)^2}}(-(c^2/g)[1-\sqrt{1-(V/c)^2}],V/g). \tag{21}$$

Each clock has the same reading when it halts acceleration. Thus when the left-hand clock halts, the right-hand clock will have ticked off an additional time of

$$t'_B-t'_A=\frac{V\ell_i/c^2}{\sqrt{1-(V/c)^2}}=\frac{L_0V}{c^2}. \tag{22}$$

This is the quantitative verification that the qualitative "master time keeper" scenario of Sec. I does indeed provide the proper amount of desynchronization.

Furthermore, after both clocks halt the distance between them in their own reference frame (the proper distance) is

$$x'_A-x'_B=L_0=\frac{\ell_i}{\sqrt{1-(V/c)^2}}, \tag{23}$$

which is longer than the (length contracted) distance ℓ_i between them in the lab frame (the frame where the two clocks move at velocity V). As described in Sec. I, the distance between clocks stretches in the proper frame and remains constant in the lab frame.

C. Co-moving frame of the right-hand clock

We are especially interested in the situation where both clocks are accelerating and the right-hand clock is at rest [for example, the first three drawings in Fig. 1(b)]. In this situation, how far apart are the two clocks? The right-hand clock is at rest, but the left-hand clock moves left. What is its velocity? What does the left-hand clock read when the right-hand clock reads τ_R? In this section, we use the notation $\beta=V/c$ and $\alpha_i=g\ell_i/c^2$.

According to Eq. (12), the coordinates for event A (right-hand clock at rest) in Fig. 4 are

$$(x'_A,t'_A)=\frac{1}{\sqrt{1-\beta^2}}(\ell_i-(c^2/g)[1-\sqrt{1-\beta^2}],(V/g)[1-\alpha_i]), \tag{24}$$

while the coordinates for event B (left-hand clock at rest) are

$$(x'_B,t'_B)=\frac{1}{\sqrt{1-\beta^2}}(-(c^2/g)[1-\sqrt{1-\beta^2}],V/g). \tag{25}$$

To find the coordinates of event C (left-hand clock while right-hand clock is at rest), we note that the shape of the

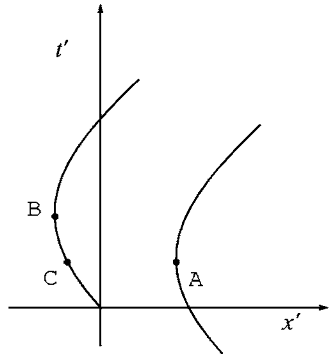

Fig. 4. Worldlines of two clocks in frame S'. The right-hand clock is at rest at event A, the left-hand clock is at rest at event B.

worldline is given through Eq. (13). The deviation of event C from its stillness event B is

$$t'_C - t'_B = t'_A - t'_B = \frac{-\alpha_i V/g}{\sqrt{1-\beta^2}}. \tag{26}$$

Using this result for $(t'-t'_0)$ in Eq. (13) gives

$$x'_C - x'_B = \frac{c^2}{g}\left[\sqrt{\frac{1-\beta^2+\alpha_i^2\beta^2}{1-\beta^2}} - 1\right], \tag{27}$$

whence

$$(x'_C, t'_C) = \frac{1}{\sqrt{1-\beta^2}}(-(c^2/g)[1-\sqrt{1-\beta^2+\alpha_i^2\beta^2}], \\ (V/g)[1-\alpha_i]). \tag{28}$$

From the coordinates of all three events, we find the distance between clocks when the right-hand clock is at rest:

$$x'_A - x'_C = \frac{c^2}{g}\left[1-\sqrt{\frac{1-(V/c)^2+(g\ell_i/c^2)^2(V/c)^2}{1-(V/c)^2}}\right] \\ + \frac{\ell_i}{\sqrt{1-(V/c)^2}}. \tag{29}$$

The velocity of the left-hand clock when the right-hand clock is at rest is obtained similarly by substituting Eq. (26) into Eq. (5). The result is

$$\frac{-V(g\ell_i/c^2)}{\sqrt{1-(V/c)^2+(g\ell_i/c^2)^2(V/c)^2}}. \tag{30}$$

We can relate the clock reading at event A (call it τ_R) to the clock reading at event C (call it τ_L) as follows. First, use the Lorentz transformation to find that the time for event C in the lab frame is

$$t_C = \frac{V}{g}\frac{\sqrt{1-\beta^2+\alpha_i^2\beta^2}-\alpha_i}{1-\beta^2}. \tag{31}$$

Then, when the right-hand clock reads τ_R, (a) the time in the lab frame is t_A given through $gt_A/c=\sinh(g\tau_R/c)$, (b) the velocity of frame S' relative to the lab is $V = gt_A/\sqrt{1+(gt_A/c)^2} = c\tanh(g\tau_R/c)$, and (c) the time for event C in the lab frame is given through Eq. (31). From these results you can work backward through $gt_C/c = \sinh(g\tau_L/c)$ to find τ_L. Executing this procedure one finds

$$\beta = \tanh(g\tau_R/c), \tag{32}$$

where

$$\sinh(g\tau_L/c) = \beta\frac{\sqrt{1-\beta^2+\alpha_i^2\beta^2}-\alpha_i}{1-\beta^2} \tag{33}$$

$$= \sinh(g\tau_R/c)[\sqrt{1+\alpha_i^2\sinh^2(g\tau_R/c)} \\ -\alpha_i\cosh(g\tau_R/c)]. \tag{34}$$

In the limit of infinitesimal τ_R and τ_L, Eq. (34) becomes

$$d\tau_L = (1-g\,\ell/c^2)d\tau_R, \tag{35}$$

the time-dilation formula of general relativity.

Although the mathematics in this section has assumed a constant proper acceleration, it is clear that the results at the halt of acceleration hold for any acceleration process in which both clocks undergo identical procedures. [That is, identical according to the lab frame, or identical according to the clocks' own timekeeping. See the discussion following Eq. (19).]

Although this discussion has yielded some insight, and shown how to construct all the numbers in Fig. 1, it hasn't really answered the question "How do two moving clocks fall out of synch?" because everything derived here assumed the Lorentz transformation from the start.

III. ACCELERATION PROCESS: GENERAL-RELATIVISTIC APPROACH

From the right-hand clock's perspective, the acceleration process has two distinct portions. At first both clocks accelerate. In the co-moving frame of the right-hand clock, the left-hand clock moves toward the left. Then the right-hand clock halts its acceleration (event A in Fig. 3). In the second portion the right-hand clock isn't accelerating (so its frame is inertial), but the left-hand clock still moves toward the left. Then the left-hand clock halts its acceleration (event B in Fig. 3). In the first portion the left-hand clock ticks slowly due to both special- and general-relativistic time dilation. In the second portion the left-hand clock ticks slowly due only to special-relativistic time dilation.

In the first portion, when both special- and general-relativistic time dilation are in play, the times ticked off on the left- and right-hand clocks are related through

$$d\tau_L = \sqrt{1-(v/c)^2}(1-g\,\ell/c^2)d\tau_R. \tag{36}$$

Here g is the acceleration of the clocks in the instantaneously co-moving inertial frame of the right-hand clock, ℓ is the distance between the clocks in that frame, and v is the velocity of the left-hand clock in that frame. Equation (36) is a natural generalization of the standard expressions for special-relativistic and for general-relativistic time dilation. A rigorous derivation is presented in Appendix A.

Substituting v from Eq. (30) and ℓ from Eq. (29) into Eq. (36) produces

$$d\tau_L = \sqrt{1-\frac{\beta^2\alpha_i^2}{1-\beta^2+\alpha_i^2\beta^2}} \\ \times\left(1-\left[1-\sqrt{\frac{1-\beta^2+\alpha_i^2\beta^2}{1-\beta^2}}\right]-\frac{\alpha_i}{\sqrt{1-\beta^2}}\right)d\tau_R \tag{37a}$$

$$= \left(1 - \frac{\alpha_i}{\sqrt{1-\beta^2+\alpha_i^2\beta^2}}\right) d\tau_R. \tag{37b}$$

But, using Eq. (5) with $t_0=0$, we obtain

$$d\tau_R = \sqrt{1-\beta^2}dt = \frac{c}{g}\frac{d\beta}{1-\beta^2}, \tag{38}$$

and so

$$d\tau_L = \frac{c}{g}\frac{\sqrt{1-\beta^2+\alpha_i^2\beta^2}-\alpha_i}{(1-\beta^2)\sqrt{1-\beta^2+\alpha_i^2\beta^2}}d\beta. \tag{39}$$

Fortunately, we have already integrated this differential equation on physical grounds: the result is Eq. (33), namely

$$\sinh\left(\frac{g\tau_L}{c}\right) = \beta\left(\frac{\sqrt{1-\beta^2+\alpha_i^2\beta^2}-\alpha_i}{1-\beta^2}\right). \tag{40}$$

It is tedious but straightforward to verify that this result satisfies the differential equation (39).

Equation (40) gives the reading on the left-hand clock in the frame where the right-hand clock is at rest, as a function of the speed of the right-hand clock in the laboratory frame. It applies to the first portion of the acceleration process.

In the second portion of the acceleration process the local frame acceleration is zero. During this portion the time t', reckoned in the frame with $V=v_f$, goes from t'_A to t'_B, as determined through Eqs. (20) and (21). Using Eq. (15) for v_f shows that during this portion the time t' goes

$$\text{from } t_f[1-\alpha_i] \text{ to } t_f. \tag{41}$$

During this portion the velocity of the left-hand clock is, from Eq. (5),

$$\frac{g(t'-t_f)}{\sqrt{1+(g(t'-t_f)/c)^2}}. \tag{42}$$

(The quantity $t'-t_f$ is negative, so this velocity is negative as well.)

The time ticked off by the moving left-hand clock during this portion is given through

$$d\tau_L = \sqrt{1-(v/c)^2}dt' = \frac{1}{\sqrt{1+(g(t'-t_f)/c)^2}}dt'. \tag{43}$$

Integrating between the limits established in Eq. (41) results in

$$\Delta\tau_L = \frac{c}{g}\,\text{arcsinh}\left(\frac{\alpha_i g t_f}{c}\right) = \frac{c}{g}\,\text{arcsinh}\left(\frac{\alpha_i\beta_f}{\sqrt{1-\beta_f^2}}\right). \tag{44}$$

Thus, at the end of the first portion, the left-hand clock reads τ_L where

$$\sinh\left(\frac{g\tau_L}{c}\right) = \beta_f\left(\frac{\sqrt{1-\beta_f^2+\alpha_i^2\beta_f^2}-\alpha_i}{1-\beta_f^2}\right). \tag{45}$$

During the second portion, the left-hand clock increments by $\Delta\tau_L$ where

$$\sinh\left(\frac{g\Delta\tau_L}{c}\right) = \frac{\alpha_i\beta_f}{\sqrt{1-\beta_f^2}}. \tag{46}$$

Adding together these two times should [see Eqs. (19) and (5)] result in a final time reading of

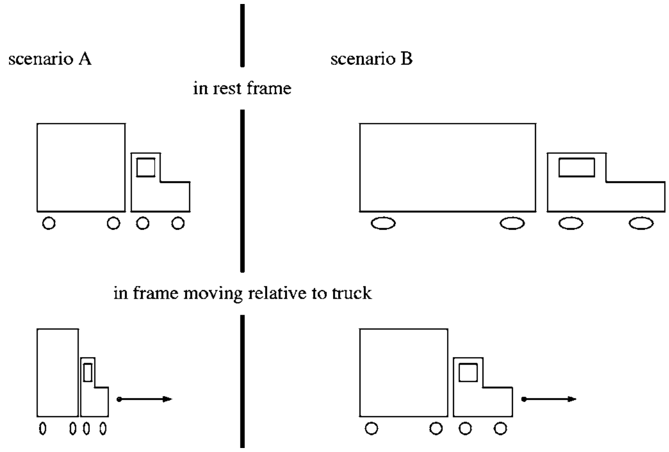

Fig. 5. On the left, a square truck is contracted in the moving frame. On the right, a square truck is stretched out in its rest frame. Both scenarios are consistent with conventional length contraction. Suppose the truck is square at rest before it executes its acceleration. If each piece of the truck executes an identical acceleration program, then the right-hand scenario applies. If each piece of the truck is accelerated by identical taps, simultaneous in the current rest frame of the truck, then the left-hand scenario applies.

$$\sinh\left(\frac{g\tau_f}{c}\right) = \frac{\beta_f}{\sqrt{1-\beta_f^2}}. \tag{47}$$

You can verify this claim (it's tedious) using the relation

$$\sinh(x+y) = \sinh(x)\sqrt{1+\sinh^2(y)} + \sinh(y)\sqrt{1+\sinh^2(x)}. \tag{48}$$

IV. THE CHARACTER OF LENGTH CONTRACTION

We have established that if two clocks, at rest in the lab frame and separated by distance ℓ_i, carry out identical acceleration programs [identical in the lab frame or identical as determined by their own clocks, see the discussion following Eq. (19)], then at the end of the program the clocks remain separated by distance ℓ_i in the lab frame, but are separated by a longer distance $\ell_i/\sqrt{1-(V/c)^2}$ in their own rest frame.

The two clocks might be in identically programmed spaceships, but they might also reside at the nose and tail of a truck. Thus if all parts of a truck undergo identical acceleration programs, the truck's proper length stretches as it accelerates. (The two clocks might also be located on two adjacent atoms, or at the right and left edges of single atom, whence all the atomic spacings and all the atoms within the truck stretch by the same factor.)

This conclusion is consistent with the conventional length contraction result. Conventional length contraction compares the length of a truck in its own frame (proper length L_0) to the shorter length $[L_0\sqrt{1-(V/c)^2}]$ of that truck in a frame in which the truck moves with velocity V. Conventional length contraction says nothing about the length of a truck before and after undergoing acceleration. Both scenarios in Fig. 5 are consistent with conventional length contraction.

Although "identical acceleration programs for each piece of truck" is a natural acceleration method, it is not the only possible nor even the only natural method. Edwin F. Taylor and A. P. French[10] have demonstrated that if each part of the truck is accelerated by taps that are simultaneous in the rest frame of the truck,[11] then the truck emerges from the acceleration process with an unchanged proper length and hence a shrunken laboratory length. Indeed, with various acceleration programs, the truck might end up with any length at all! (One such program is discussed in Problem 2 of Appendix B.) As J. S. Bell has phrased it,[12] "A system set brutally in motion may be bruised, or broken, or heated, or burned." Any truck that really did accelerate from 0 to $0.9c$ in 30 s, as represented in Fig. 1, would likely end up with a proper length thinner than photographic film.

So what acceleration method is most likely to be employed in nature? The answer comes through the realization that, in scenario B of Fig. 5, the separation between atoms stretches in the rest frame. This separation length is dictated by the attractions and repulsions between atoms, which do not change in the proper frame. A similar conclusion can be made for the shape of an atom. (Bell analyzed a classical model of the atom attesting to this conclusion.[12]) Long before the atoms stretch to twice their equilibrium separations, they will snap back. This shifting of atoms might, depending on the material and the acceleration program, be accompanied by macroscopic vibrations. But after the vibrations have died out, the truck's proper length will be the same before and after acceleration.

Although any acceleration method, and hence any final length, is *possible*, the most natural acceleration method is one that results in the conventional scenario A, such as the Taylor-French method.

V. THE THREAD-BETWEEN-SPACESHIPS PARADOX

The thread-between-spaceships paradox was invented by Edmond Dewan and M. Beran[13] in 1959, but is often called the "Bell spaceship paradox" because it was popularized by Bell[12] in 1976. Suppose a thread is stretched taunt between the two clocks before they start their identical acceleration programs. As the clocks speed up to relativistic velocities, will the thread break? This simple question has resulted in some controversy.[14]

The taunt thread is like the truck discussed in Sec. IV. Although nothing in relativity *prohibits* all parts of the thread from undergoing identical acceleration programs (thus maintaining the same length in the lab frame and stretching in the proper frame), this acceleration method is unlikely to be realized with a real thread made of atoms. The real thread will likely contract in the lab frame and maintain its same proper length. When it does so, the thread will break. (In the lab frame, the clocks maintain the same separation while the thread contracts, leading to breakage. In the frame of the right-hand clock, the clocks move apart while the thread maintains the same length, leading to breakage.)

Robert Firth[15] produced a simple and superficially convincing argument against this resolution of the thread-between-spaceships paradox. According to the principle of equivalence, a thread stretched taunt between two accelerating spaceships is equivalent to a thread stretched taunt vertically in gravity. In the latter case the thread doesn't break, so in the former case it won't either.

Quantitatively, suppose the thread is strong enough that it can be stretched to twice its natural length before snapping. If the thread-between-spaceships really does break, then it will break when the spaceships attain the speed $(\sqrt{3}/2)c$. According to Eq. (5), this speed occurs after the spaceships have been accelerating for a lab time of $\sqrt{3}c/g$ or a spaceship time of (c/g) arcsinh $(\sqrt{3})$. For $g=9.8\ \mathrm{m/s^2}$, the spaceship time required is 40.3 million seconds or 466 days. Thus any thread left hanging on the Earth's surface for two years will necessarily break. Many flags, draperies, and trousers have been left hanging near the Earth's surface for this long without breaking.

The error in this appealing argument is that the principle of equivalence applies only to "small enough" reference frames for "short enough" times.[16] The quantitative requirement for applicability is that $2\Delta\Phi/c^2 \ll 1$, where Φ is the gravitational potential, which in this case is $\Delta\Phi = g\Delta z$. If the trousers fall from rest, then by classical energy conservation $g\Delta z \approx v^2/2$, so the applicability requirement is that $(v/c)^2 \ll 1$. In other words, just as length contraction becomes significant, the equivalence principle becomes inapplicable. If trousers fall for two years, this time is not "short enough" for the equivalence principle to apply.

VI. THE TWIN PARADOX

The twin paradox is much loved and well studied.[17] The statement and resolution of the paradox, in purely special-relativistic terms, is given in the following.

A traveler journeys at speed $\mathcal{V}$ from the Earth to a star located (in the Earth's frame) a distance L_0 away. Then—after a brief time interval to turn around at the star—she returns to Earth at the same speed. In the Earth's frame this journey requires time $2L_0/\mathcal{V}$, but because of special-relativistic time dilation the traveler's clock ticks slowly and returns having ticked off a smaller time $\sqrt{1-(\mathcal{V}/c)^2}(2L_0/\mathcal{V})$.

From the traveler's point of view the star journeys toward her at speed $\mathcal{V}$ over a contracted length of $\sqrt{1-(\mathcal{V}/c)^2}L_0$, so the star reaches her after time $\sqrt{1-(\mathcal{V}/c)^2}L_0/\mathcal{V}$. (And sure enough this is the amount of time ticked off during the outbound leg by the traveler's clock.) During this outbound leg the moving Earth clock ticks off a smaller time $[1-(\mathcal{V}/c)^2]L_0/\mathcal{V}$. Similarly for the return leg. But during the turnaround the Earth clock changes from being the front clock, set behind the star clock by $L_0\mathcal{V}/c^2$, to being the rear clock, set ahead of the star clock by $L_0\mathcal{V}/c^2$. That is, during the brief turnaround interval, the Earth clock has (to the traveler) advanced by $2L_0\mathcal{V}/c^2$. The total time ticked off by the Earth clock is thus

$$[1-(\mathcal{V}/c)^2]L_0/\mathcal{V} + 2L_0\mathcal{V}/c^2 + [1-(\mathcal{V}/c)^2]L_0/\mathcal{V} = 2L_0/\mathcal{V}. \tag{49}$$

The two points of view obtain identical results, as they must. But they explain these results in different ways: From the Earth's point of view the traveler's clock ticks slowly. From the traveler's point of view the Earth clock usually ticks slowly, but it jumps ahead rapidly during the brief turnaround interval.

This resolution of the paradox is correct, but it leaves most students with a gnawing pit in their guts. How, to the traveler, can the Earth clock advance so rapidly during that brief turnaround interval? Within special relativity one can only say, "That's an accelerated reference frame, so I can tell you the result at the end of the acceleration but I have to be silent about what's going on during acceleration." Within general relativity there's a more satisfactory answer. During the turn-

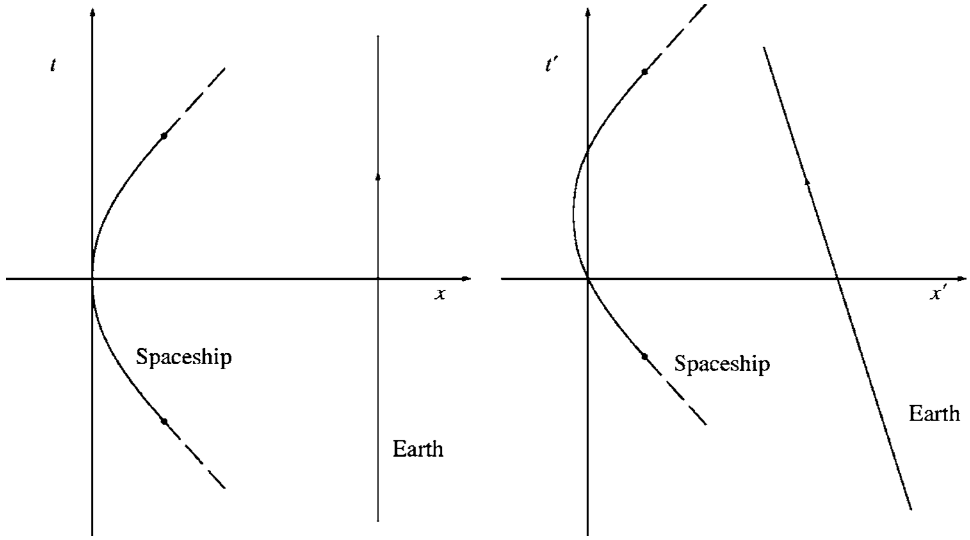

Fig. 6. The traveler's turnaround, as observed (left) in the Earth's frame and (right) in a frame moving right at velocity V relative to the Earth.

around, the traveler's clock is a "lower" clock while the Earth clock is a "higher" clock. The full quantitative analysis below is long, but the idea that the high Earth clock ticks off more time than the low traveler's clock is straightforward general-relativistic time dilation.

By using the machinery of Sec. II, it is not difficult to show that in the Earth's reference frame, the turnaround illustrated in Fig. 6 requires time

$$t_T = \frac{2\mathcal{V}/g}{\sqrt{1-(\mathcal{V}/c)^2}}. \tag{50}$$

In the reference frame moving at velocity V relative to the Earth, the Earth's worldline is

$$x' = -Vt' + L_0\sqrt{1-(V/c)^2} \tag{51}$$

and, at the instant the spaceship is at rest, the distance to the Earth is

$$\frac{c^2}{g}\left(\frac{1+gL_0/c^2}{\sqrt{1-(V/c)^2}} - 1\right). \tag{52}$$

An analysis very similar to that of Appendix A produces a very similar time dilation result: In the co-moving frame of the spaceship, the spaceship clock ticks off time $d\tau_{SS}$ while the Earth clock ticks off time $d\tau_E$, where

$$d\tau_E = \sqrt{1-(v_E/c)^2}(1+g\,\ell\,/c^2)d\tau_{SS}. \tag{53}$$

Here ℓ is the distance from the Earth to the spaceship in the co-moving frame of the spaceship. From Eq. (52),

$$d\tau_E = \sqrt{1-(V/c)^2}\left(1+\left[\frac{1+gL_0/c^2}{\sqrt{1-(V/c)^2}} - 1\right]\right)d\tau_{SS} \tag{54a}$$

$$= (1+gL_0/c^2)d\tau_{SS}. \tag{54b}$$

Integrate Eq. (54b) to find

$$\Delta\tau_E = \Delta\tau_{SS} + \frac{gL_0}{c^2}\Delta\tau_{SS}. \tag{55}$$

We now know that the Earth clock ticks off more time than the spaceship clock, the only question is how much more.

The time ticked off by the spaceship during turnaround is related to the lab time elapsed during turnaround through

$$\frac{g(t_T/2)}{c} = \sinh\left(\frac{g(\Delta\tau_{SS}/2)}{c}\right), \tag{56}$$

but Eq. (50) tells us that (notice that the following results are independent of g and t_T)

$$\Delta\tau_E = \Delta\tau_{SS} + \frac{2L_0}{c}\,\text{arcsinh}\left(\frac{\mathcal{V}/c}{\sqrt{1-(\mathcal{V}/c)^2}}\right) \tag{57a}$$

$$= \Delta\tau_{SS} + \frac{L_0}{c}\ln\left(\frac{1+\mathcal{V}/c}{1-\mathcal{V}/c}\right). \tag{57b}$$

If, as postulated, the turnaround time is small, then $\mathcal{V}/c$ is small and the logarithm in Eq. (57b) is close to $2\mathcal{V}/c$. Thus the general relativistic result is that, for short turnaround times,

$$\Delta\tau_E = \Delta\tau_{SS} + \frac{2L_0\mathcal{V}}{c^2}, \tag{58}$$

in agreement with the special relativistic result.

ACKNOWLEDGMENTS

I was able to pursue this question only because of a research status leave from Oberlin College. Stephen P. Boughn, James B. Hartle, N. David Mermin, Edwin F. Taylor, and two anonymous reviewers graciously examined drafts of this paper and suggested improvements, saving me from a number of embarrassments. Remaining flaws are, of course, my own.

APPENDIX A: DERIVATION OF GENERAL-RELATIVISTIC TIME DILATION

We derived the standard formula for general-relativistic time dilation at Eq. (33). Here we use the same procedure to derive the formula for the case where the two clocks are not only accelerating, but the left-hand clock is moving as well.

In this case the right-hand clock starts from rest, so its trajectory has [see Eqs. (5) and (7)] $x_0=\ell$ and $t_0=0$. The left-hand clock starts with velocity v, so its trajectory has $x_0=(c^2/g)[1-\gamma]$ and $t_0=-\gamma v/g$, where $\gamma=1/\sqrt{1-(v/c)^2}$. We again use $\alpha=g\ell/c^2$ and $\beta=V/c$.

According to Eq. (12), the coordinates in frame S' of the events A and B in Fig. 4 are for event A (right-hand clock at rest)

$$(x'_A,t'_A)=\frac{1}{\sqrt{1-\beta^2}}((c^2/g)[\alpha-1+\sqrt{1-\beta^2}], (V/g)[1-\alpha]), \tag{A1}$$

and for event B (left-hand clock at rest)

$$(x'_B,t'_B)=\frac{1}{\sqrt{1-\beta^2}}((c^2/g)\gamma[Vv/c^2-1] + (c^2/g)\sqrt{1-\beta^2},(\gamma/g)[V-v]). \tag{A2}$$

Thus

$$t'_C-t'_B=t'_A-t'_B=\frac{-1/g}{\sqrt{1-\beta^2}}[V(\gamma+\alpha-1)-\gamma v], \tag{A3}$$

so

$$x'_C-x'_B=\frac{c^2}{g}\left[\sqrt{1+\left(\frac{g(t'_A-t'_B)}{c}\right)^2}-1\right] \tag{A4}$$

$$=\frac{c^2/g}{\sqrt{1-\beta^2}}[\sqrt{1-\beta^2+[\beta(\gamma+\alpha-1)-\gamma v/c]^2} - \sqrt{1-\beta^2}]. \tag{A5}$$

Thus the coordinates in frame S' of event C (left-hand clock moves left, at same instant that right-hand clock is at rest) are

$$x'_C=\frac{c^2/g}{\sqrt{1-\beta^2}}[\gamma(Vv/c^2-1) + \sqrt{1-\beta^2+[\beta(\gamma+\alpha-1)-\gamma v/c]^2}], \tag{A6}$$

$$t'_C=\frac{V/g}{\sqrt{1-\beta^2}}[1-\alpha], \tag{A7}$$

and in the lab frame

$$t_C=\frac{V/g}{1-\beta^2}[1-\alpha+\gamma(Vv/c^2-1) + \sqrt{1-\beta^2+[\beta(\gamma+\alpha-1)-\gamma v/c]^2}]. \tag{A8}$$

Now we're prepared to derive time dilation, as at Eq. (33), except that now we need only the case in which τ_R is an infinitesimal. When the right-hand clock reads $d\tau_R$, (a) the time in the lab frame is $dt_A=d\tau_R$, (b) the velocity of frame S' relative to the lab is gdt_A, and (c) the time for event C in the lab frame is $dt_C=dt_A(1-\alpha)$ because all the β's are infinitesimal. From these results you can work backward through $dt_C=\gamma d\tau_L$ to find

$$d\tau_L=\sqrt{1-(v/c)^2}(1-g\ell/c^2)d\tau_R. \tag{A9}$$

APPENDIX B: SUGGESTED PROBLEMS

Here are two problems that can be assigned to students to help drive home the ideas presented in this paper, and one research question.

(1) Verify the distances, speeds, and clock readings shown in Fig. 1. For example, show that the bottom right clock reads 31.71 s.

(2) Suppose the two clocks commence acceleration simultaneously in the lab frame, and halt acceleration simultaneously in the frame moving relative to the lab at $V/c=\beta_f$, when the left-hand clock is stationary in that frame. Show that when they halt in this frame, the right-hand clock is moving toward the right with velocity $v_f\alpha_i/\sqrt{1+\alpha_i^2\beta_f^2}$, and that the distance between them is

$$\frac{1}{\sqrt{1-\beta_f^2}}((c^2/g)\sqrt{1-\beta_f^2+\alpha_i^2\beta_f^2}-(c^2/g)\sqrt{1-\beta_f^2}+\ell_i). \tag{B1}$$

(3) Section II shows that at the end of the acceleration program, the two clocks (in their own frame) are separated by the distance $L_0=\ell_i/\sqrt{1-(V/c)^2}$ and have time readings that differ by L_0V/c^2. Section III explains this time difference through general-relativistic time dilation, assuming constant proper acceleration. The results of Sec. II must apply even when the acceleration is not uniform, but I have not been able to find the correct generalization. Can you? (This research question should not be inflicted on a typical student.)

[a)]Electronic address: Dan.Styer@oberlin.edu

[1]E. F. Taylor and J. A. Wheeler, *Spacetime Physics* 2nd ed. (Freeman, New York, 1992).

[2]Rachel E. Scherr, Peter S. Shaffer, and Stamatis Vokos, "The challenge of changing deeply held student beliefs about the relativity of simultaneity," Am. J. Phys. **70**(12), 1238–1248 (2002).

[3]James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, San Francisco, 2003).

[4]Albert Einstein, "Zur Elektrodynamik bewegter Körper," Ann. Phys. **17**, 891–921 (1905); translated as "On the electrodynamics of moving bodies" in *The Principle of Relativity* (Dover, New York, 1923), p. 48. The overgeneralization of this result to "before and after acceleration" appears already on p. 41 of this paper!

[5]This is the very first thing I ever heard about relativity—a substitute teacher told it to my eighth-grade science class at Neshaminy Junior High School—and I had mixed emotions upon discovering that it leaves out important facets of the issue.

[6]J. Bailey, K. Borer, F. Combley, H. Drumm, F. Krienen, F. Lange, E. Picasso, W. von Ruden, F. J. M. Farley, J. H. Field, W. Flegel, and P. M. Hattersley, "Measurements of relativistic time dilatation for positive and negative muons in a circular orbit," Nature (London) **268**, 301–305 (1977); Ref. 3, p. 64.

[7]See, for example, Robert Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), p. 84, or Paul Lorrain and Dale Corson, *Electromagnetic Fields and Waves*, 2nd ed. (Freeman, San Francisco, 1970), p. 217. See also Ref. 4, p. 63.

[8]R. H. Good, "Uniformly accelerated reference frame and twin paradox," Am. J. Phys. **50**(3), 232–238 (1982).

[9]S. P. Boughn, "The case of identically accelerated twins," Am. J. Phys. **57**(9), 791–793 (1989); Ref. 1, pp. 117–118.

[10]Edwin F. Taylor and A. P. French, "Limitation on proper length in special relativity," Am. J. Phys. **51**(10), 889–893 (1983); Ref. 1, pp. 119–120; Ref. 3, Prob. 6-7, p. 132.

[11] We saw following Eq. (19) that, if the acceleration program is identical for each piece of the truck, then there is no rest frame for the whole truck. But under the Taylor-French scheme of simultaneous taps, there is such a frame.

[12] John S. Bell, "How to teach special relativity," Prog. Scientific Culture **1**(2), 1–13 (1976); reprinted in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U. P., Cambridge, 1987), pp. 67–80. See particularly p. 8 (reprint p. 75).

[13] E. Dewan and M. Beran, "Note on stress effects due to relativistic contraction," Am. J. Phys. **27**(10), 517–518 (1959).

[14] Arthur A. Evett and Roald K. Wangsness, "Note on the separation of relativistically moving rockets," Am. J. Phys. **28**(9), 566 (1960); Paul J. Nawrocki, "Stress effects due to relativistic contraction," Am. J. Phys. **30**(10), 771–772 (1962); Edmond M. Dewan, "Stress effects due to Lorentz contraction," Am. J. Phys. **31**(5), 383–386 (1963); J. E. Romain, "A geometrical approach to relativistic paradoxes," Am. J. Phys. **31**(8), 576–585 (1963); Arthur A. Evett, "A relativistic rocket discussion problem," Am. J. Phys. **40**(8), 1170–1171 (1972); S. S. Gershteĭn and A. A. Logunov, "J. S. Bell's problem," Phys. Part. Nucl. **29**, 463–468 (1998); G. Cavalleri and E. Tonni, "Comment on 'Čerenkov effect and the Lorentz contraction'," Phys. Rev. A **61**, 026101-1–2 (2000); D. T. Cornwell, "Forces due to contraction on a cord spanning between two spaceships," Europhys. Lett. **71**, 699–704 (2005).

[15] Robert Firth, newsgroup sci.physics, 22 September 1993.

[16] Reference 1, pp. 30–34; Ref. 3, pp. 119 and 131.

[17] Paul Langevin, "L'évolution de l'espace et du temps," Scientia **X**, 31–54 (1911); Gerald Holton, "Resource letter SRT 1 on special relativity theory," Am. J. Phys. **30**(6), 462–469 (1962); Talal A. Debs and Michael L. G. Redhead, "The 'twin paradox' and the conventionality of simultaneity," Am. J. Phys. **64**(4), 384–392 (1996); E. Minguzzi, "Differential aging from acceleration: An explicit formula," Am. J. Phys. **73**(9), 876–880 (2005).