Relativistic transformations of light power

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(Received 21 August 1978; accepted 16 February 1979)

Using a photon-counting technique, we find the angular distribution of emitted and detected power and the total radiated power of an arbitrary moving source. We compare these expressions to the incorrect or incomplete expressions previously reported. We use the technique to verify the predicted effect of the earth's motion through the cosmic blackbody radiation.

INTRODUCTION

In the course of a recent investigation[1] on the appearance of stars from a relativistic spaceship, we came to realize that the transformation of the intensity of light is a little appreciated aspect of special relativity theory. Of the many references[2] on the visual appearance of rapidly moving objects, the only one[3] which treated intensity at all turned out to be incorrect. The widely used textbooks on relativistic electrodynamics[4,5] do not address the problem. Moreover, one of them repeats an incomplete statement[6] on the transformation of total radiated power which could mislead an unwary reader to use the conclusion in a situation where it does not apply. These errors and misconceptions have propagated.[7]

In this investigation a technique of counting photons is used to develop the transformations for the angular distributions of emitted and detected power, and for the total emitted power. These transformations are then related to one another and to the transformation for the intensity of plane waves, and compared to the incomplete or incorrect transformations previously known. An example which shows the distinction between the power absorbed by a fixed detector and the power emitted by a moving source is presented in an appendix. In another appendix the photon counting technique is used to recalculate the effect of the earth's motion through the cosmic blackbody radiation at ~3 K.

DERIVATION OF TRANSFORMATIONS

Let us define the reference frames to be used. Frame S_0 (proper frame) is defined by the source and is not necessarily inertial. Frame S' (instantaneous proper frame) is inertial and instantaneously coincides with frame S_0. Frame S (frame of observation) is an inertial frame relative to which the source is moving at velocity **v**. The usual Lorentz transformation along the z-z' axes connects frame S and frame S' (or S_0).

Consider a photon which has propagation vector $(k'; \mathbf{k}') = (k'; k'\sin\theta'\cos\phi', k'\sin\theta'\sin\phi', k'\cos\theta')$ in frame S'. In frame S the same expression applies except with all primes removed. The usual expressions for aberration and angle-dependent Doppler effect connect the two representations. We shall need the following:

$$k = Dk' = \gamma(1 + \beta\cos\theta')k', \tag{1}$$

$$\cos\theta = (\cos\theta' + \beta)/(1 + \beta\cos\theta'), \tag{2}$$

$$\phi = \phi', \tag{3}$$

$$\cos\theta' = (\cos\theta - \beta)/(1 - \beta\cos\theta), \tag{4}$$

$$D = \gamma^{-1}(1 - \beta\cos\theta)^{-1}. \tag{5}$$

The last two of these are found from the inverse of the Lorentz transformation which gives the first two. A group of photons may be associated with the solid angle $d\Omega' = d(\cos\theta')d\phi'$ in frame S' and with $d\Omega = d(\cos\theta)d\phi$ in frame S. The connection between the two, found by differentiating Eqs. (2) and (3), is

$$d\Omega = (1 - \beta^2)(1 + \beta\cos\theta')^{-2}\, d\Omega' = D^{-2}\, d\Omega'. \tag{6}$$

Now let us describe the emission of radiation by the source in terms of photons. In frame S' or S_0, the number of photons emitted during time dt' and having propagation vector in the increment $dk'd\Omega'$ is given by

$$dN = n'(\mathbf{k}')dk'd\Omega'dt'.$$

When we examine these same photons in frame S, we see that they are emitted during time $dt = \gamma dt'$ (time dilatation) and into the increment $dkd\Omega$. Since it is same number of photons in either frame, we write

$$dN = n(\mathbf{k})\, dkd\Omega dt.$$

Upon inserting the transformations of all the differentials, we find the transformation of the rate of emission of photons

$$n(\mathbf{k}) = n'(\mathbf{k}')D/\gamma. \tag{7}$$

To find the angular distribution of radiated power, multiply by the photon energy $\hbar ck$ and integrate over all wave numbers

$$P(\hat{k}) = \int_0^\infty \hbar ckn(\mathbf{k})dk = \left(\frac{D^3}{\gamma}\right)\int_0^\infty \hbar ck'n'(\mathbf{k}')dk' = \left(\frac{D^3}{\gamma}\right)P'(\hat{k}') = D^2(1 + \beta\cos\theta')P'(\hat{k}'). \tag{8}$$

The luminosity of the source is the total power emitted into all solid angle

$$L = \int_{4\pi} P(\hat{k})d\Omega = \int_{4\pi} P'(\hat{k}')(1 + \beta\cos\theta')d\Omega' = L' + \beta\int_{4\pi}\cos\theta' P'(\hat{k}')d\Omega'. \tag{9}$$

This relationship for luminosity can be found in other ways as well. Let us consider that the system consists only of the source and its radiation. Then the source acts as a

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photon rocket and is not inertial. Its proper (rest) mass M is continuously decreasing to provide the proper luminosity

$$L_0 = L' = \frac{-d(Mc^2)}{dt_0}. \qquad (10)$$

In frame S its four-vector velocity $(u^0;\mathbf{u}) = (\gamma c;\gamma \mathbf{v})$ is not constant. The energy still in the source is $E = Mcu^0$, and the luminosity is given by

$$\begin{aligned} L &= \frac{-dE}{dt} = -\left(\frac{dt_0}{dt}\right)\left(\frac{d(Mcu^0)}{dt_0}\right) \\ &= -\left(\frac{c}{u^0}\right)\left(\frac{cu^0 dM}{dt_0} + \frac{Mc\,du^0}{dt_0}\right) \\ &= L' - \frac{(Mc^2/u^0)du^0}{dt_0}. \end{aligned} \qquad (11)$$

The second term involves the time component of the four-vector acceleration. By the Lorentz transformation we rewrite it in terms of the local acceleration $\mathbf{a}'$:

$$L = L' - Mc\boldsymbol{\beta}\cdot\mathbf{a}'. \qquad (12)$$

The equation of motion of a photon rocket can be written[8]

$$M\mathbf{a}' = -\mathbf{G}', \qquad (13)$$

where $\mathbf{G}'$ (the negative of the thrust) is the rate at which momentum is emitted into the exhaust (radiation) in the proper frame. That in turn can be expressed in terms of the rate of emission of photons

$$\mathbf{G}' = \int \hbar \mathbf{k}' n'(\mathbf{k}') dk' d\Omega' = \int_{4\pi} \hat{k}' P'(\hat{k}') c^{-1} d\Omega'. \qquad (14)$$

When this is inserted into Eq. (12) we again obtain Eq. (9).

The expression [Eq. (8)] for the angular distribution of the power emitted from a moving source is not particularly useful, since that quantity is not directly measurable. What one measures is the power received at a detector at some distance from the source, and that quantity transforms differently.

In frame S' the source is at rest at the origin as it emits radiant energy at the rate $P'(\hat{k}')d\Omega'$ into the solid angle $d\Omega'$. As the radiation propagates it stays in that solid angle. Thus the angular distribution of the detected power $Q'(\hat{k}')$ is equal to the (retarded) angular distribution of emitted power $P'(\hat{k}')$. Let us imagine that the radiation consists of radial streams of photons, and that the photons in the solid angle $d\Omega'$ are spaced at radial separation ζ' [see Fig. 1(a)]. Thus the world line of the ith photon of this stream is described by

$$\begin{aligned} x_i' &= \sin\theta' \cos\phi' (ct' - i\zeta'), \\ y_i' &= \sin\theta' \sin\phi' (ct' - i\zeta'), \\ z_i' &= \cos\theta' (ct' - i\zeta'). \end{aligned} \qquad (15)$$

When we transform to frame S, the world line of this same photon is described by

$$\begin{aligned} x_i &= \sin\theta \cos\phi (ct - \gamma i\zeta'), \\ y_i &= \sin\theta \sin\phi (ct - \gamma i\zeta'), \\ z_i &= \cos\theta (ct - \gamma i\zeta') + \gamma\beta i\zeta'. \end{aligned} \qquad (16)$$

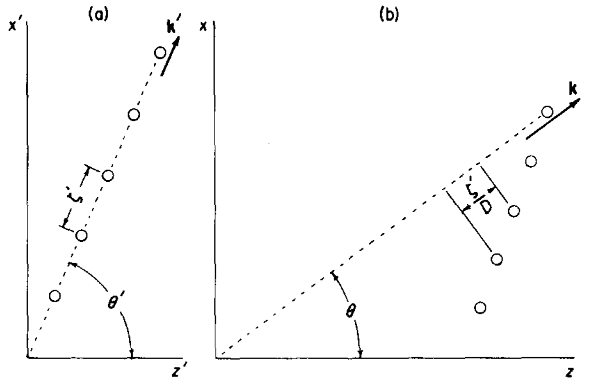

Fig. 1. (a) Stream of photons measured simultaneously in frame S', in which the source is at rest. The spacing is ζ'. (b) The same stream of photons measured simultaneously in frame S, in which the source is moving along the z axis. The spacing along the direction of propagation is ζ'/D.

This stream of photons is propagating in the aberrated direction $\hat{k}$, but they are offset in echelon because of the motion of the source in this frame [see Fig. 1(b)]. To determine their spacing along the direction of propagation, let us take the scalar product of $\mathbf{r}_i$ with the propagation vector

$$\mathbf{k}\cdot\mathbf{r}_i = k[ct - i\gamma(1 - \beta\cos\theta)\zeta'] = k(ct - i\zeta'/D). \qquad (17)$$

We identify the spacing as ζ'/D. (This same result can be found from the invariance of $kct - \mathbf{k}\cdot\mathbf{r}_i$.) Thus the flux of photons into a detector is increased by this factor D, while the energy of each photon is increased by another factor D and the stream of photons is compressed in solid angle by yet another factor D^2 [Eq. (6)]. Altogether we find that the angular distribution of detected power transforms as

$$Q(\hat{k}) = D^4 Q'(\hat{k}'). \qquad (18)$$

This is obviously quite different from the transformation of the radiated angular distribution [Eq. (8)].

For completeness, let me restate the transformation for the intensity (power per unit area) of a plane wave, or equivalently, of a beam of photons propagating parallel. For a beam of photons which transform according to Eqs. (1)-(5), the intensity transforms as

$$I = D^2 I'. \qquad (19)$$

This can be derived in several ways.[9] The method of counting photons analogous to that used here ascribes one factor D to the energy of each photon and the other factor D to the density of photons in space.

Figure 2 shows the three unique transformations [Eqs. (8), (18), and (19)].

DISCUSSION

The transformations found here for angular distributions of emitted and detected power and for luminosity [Eqs. (8), (9), and (18)] are remarkable in that they have apparently never before been presented correctly. There are incomplete or incorrect statements on luminosity and on emitted angular distribution in the literature, while the difference between emitted and detected power distributions is typically[10] attributed to the difference between the in-

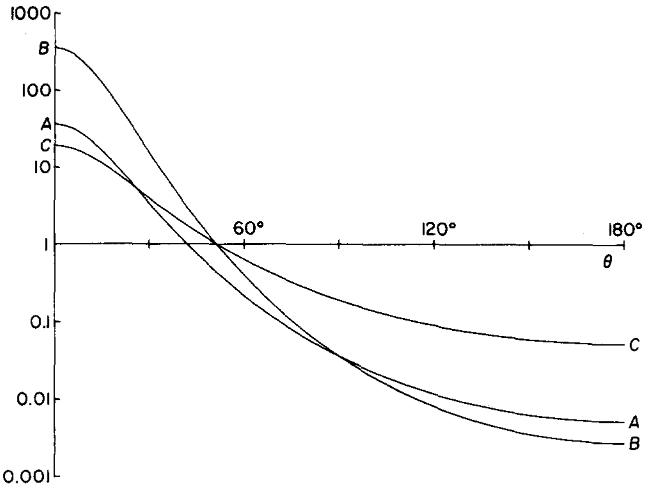

Fig. 2. Factors of power transformations as functions of colatitude in the frame of observation, for $\beta = 0.9$. Curve *A:* D^3/γ, the ratio of angular distributions of emitted power. Curve *B:* D^4, the ratio of angular distributions of detected power. Curve *C:* D^2, the ratio of intensities of a plane wave.

crement of detection time and the increment of the retarded time of emission.

There are two levels at which errors, omissions, or misconceptions have crept into previous work. At the first level is the usual argument[6,7] which leads to the conclusion that the luminosity is invariant, in apparent contradiction to Eq. (9). The syllogism can be paraphrased as follows. "Both the radiated energy and the time interval are the fourth components of four vectors and transform identically. Therefore, their ratio is invariant." The transformations can be identical only if the two complete four-vectors are parallel in every frame. The displacement between two event points on the source's world line, of which the time interval is a component, is purely timelike in the proper frame. The radiated energy momentum must be purely timelike in the same frame. That is, the radiation must be sufficiently symmetric that there is no thrust of the photon rocket. This happens to be true for the cases of greatest interest—isotropic radiation and the dipole radiation of an accelerated electron. For either of those systems the luminosity is indeed invariant. Since this condition is not true for arbitrary sources, the syllogism should be restated in a form which is generally correct.

At the second level is a logical fault which has been employed[3,7] to find a transformation of the angular distribution of emitted power, which does not agree with Eq. (8). In that backward argument it is asserted that since the luminosity is invariant [*sic*], the product $P(\hat{k})d\Omega$ must therefore be invariant. Thus one concludes $P(\hat{k}) = D^2P'(\hat{k}')$. In actuality, in order for $\int_{4\pi}P(\hat{k})d\Omega$ to be equal to $\int_{4\pi}P'(\hat{k}')d\Omega'$, it is sufficient but not necessary that $P(\hat{k})d\Omega$ be invariant. In one frame this quantity could be multiplied by any function of angle which has a weighted average value of unity, and the luminosity would still be invariant. The particular function $(1 + \beta\cos\theta')$ which was found and included in Eq. (8) has exactly unity as its weighted average value when $P'(\hat{k}')$ has sufficient symmetry.

I suspect that this misconception about the emitted angular distribution may have arisen from an analogy to certain nuclear physics experiments. When one is counting photons emitted by a moving source (e.g., $\pi^0 \rightarrow \gamma + \gamma$), the number of counts in a solid angle defined by the photons themselves is invariant. It follows that the angular distribution of counts transforms as the inverse of solid angle. However, here we are concerned with power, the rate of emission of energy. The effects of time dilatation upon rate and of the Doppler shift upon photon energy must be included.

Let us now reconcile the difference between $P(\hat{k})d\Omega$, the power radiated into solid angle $d\Omega$ by the moving source, and $Q(\hat{k})d\Omega$, the power detected by a fixed detector in the same solid angle. From Eqs. (8) and (18) plus the equality of $P'(\hat{k}')$ and $Q'(\hat{k}')$, we have

$$\begin{aligned} P(\hat{k}) - Q(\hat{k}) &= (D^3/\gamma)P'(\hat{k}') - D^4Q'(\hat{k}') \\ &= -D^4Q'(\hat{k}')(1 - 1/\gamma D) \\ &= -\beta\cos\theta Q(\hat{k}). \end{aligned} \tag{20}$$

When this is multiplied by $d\Omega$ we recognize it as the statement of energy conservation in a time-dependent volume. The volume is the cone shown in Fig. 3, with its apex at the source and its base at the detector. The first term on the left side is the power entering this volume and the second term is the power leaving it. The right side is the rate of change of the energy contained in the volume, due to the fact that the volume is changing.

The first conclusion from this investigation is that the "headlight" effect upon the angular distribution of power from a moving source is far more pronounced than predicted by Weisskopf. For a source which is isotropic in its proper frame, the detected power per unit solid angle will be given by curve B of Fig. 2. By contrast, Weisskopf stated that he was considering the emitted power (curve A) but presented an expression equivalent to curve C. This mistake in the intensity does not invalidate the main content of his paper, on the apparent rotation of a moving object. Secondly, although the argument employed by Jackson and others is incomplete, the conclusion is correct that the luminosity is invariant for the dipole radiation of an accelerated electron. All derivations based directly upon that invariance will be unaffected. Thirdly, the transformation found here for the angular distribution of detected power should allow a somewhat simplified derivation of the properties of synchrotron radiation.

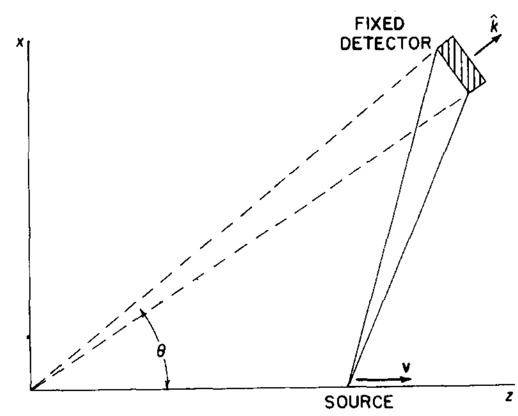

Fig. 3. Volume instantaneously occupied by photons propagating in a particular solid angle is outlined by solid lines. The solid angle is defined by dashed lines from the retarded position of the source. (Any displacement of the source due to its acceleration is ignored.)

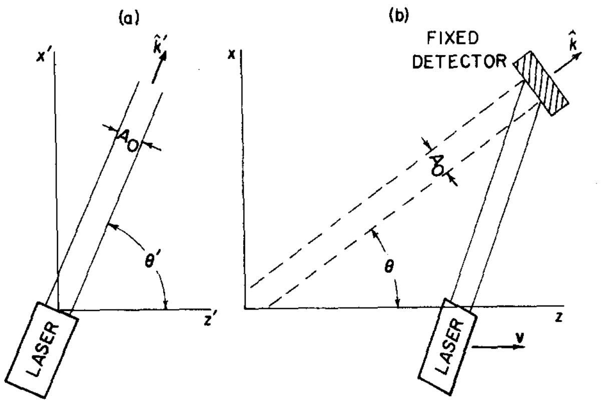

Fig. 4. (a) Laser rocket in its proper frame. The beam of area A_0 and intensity I_0 propagates at angle θ'. (b) In frame S the rocket moves along the z axis at velocity $\mathbf{v}$. The beam propagates at angle θ and has area A_0 normal to that direction (dashed lines). The volume occupied by the beam at a single instant t is shown by solid lines. (Any displacement of the rocket due to its acceleration is ignored.)

APPENDIX A

As a simple example which shows the distinction between emitted power and detected power for a moving source as well as the luminosity transformation, let us consider an idealized laser rocket. The proper luminosity L_0 is emitted as a uniform collimated beam of proper area A_0. The proper intensity of the beam is $I_0 = L_0/A_0$. In the instantaneous proper frame S' the beam is at angle θ' to the velocity $\mathbf{v}$, with which the laser and frame S' are moving relative to frame S [see Fig. 4(a)]. The Doppler factor for the transformation to frame S is $D = \gamma(1 + \beta\cos\theta')$. In frame S the beam propagates at the aberrated angle θ but the cross-sectional area normal to the direction of propagation is invariant with value A_0. The intensity measured in frame S is $I = D^2 I_0$, so that a detector which intercepts the whole beam will absorb power $IA_0 = D^2 L_0$. The beam between the laser and the detector occupies a volume which is changing at the rate $-vA_0\cos\theta$ [Fig. 4(b)]. The radiant energy in that volume is changing at the rate $(I/c) \times (-vA_0\cos\theta) = -\beta\cos\theta D^2 L_0$. The sum of this last quantity and the power absorbed in the detector is equal to the luminosity in this frame, i.e., the power emitted by the laser at the retarded time corresponding to the time of absorption

$$L = D^2L_0 - \beta\cos\theta D^2 L_0 = L_0 D/\gamma = L_0(1 + \beta\cos\theta').$$

This is equivalent to Eq. (9), when the entire emission occurs at the single angle θ'.

APPENDIX B

As another example of the technique of counting photons let us predict the effect of the earth's motion through the cosmic blackbody radiation at $\sim$3 K. The source frame S' is the proper frame of the radiation, i.e., the frame in which the radiation is isotropic. Frame S is the frame of the earth, which is moving at velocity $-\mathbf{v}$ relative to frame S'.

In frame S' the number of photons in volume dV' which have propagation vector $\mathbf{k}'$ in the increment $dk'd\Omega'$ is given by the blackbody relation

$$dN = 2^{-2}\pi^{-3}[\exp(\hbar ck'/k_BT') - 1]^{-1}\, k'^2 dk' d\Omega'\, dV',$$

where the Boltzmann constant is designated k_B to distinguish it from wavenumber. This same number of photons viewed in frame S will have propagation vector $\mathbf{k}$ in the interval $dk d\Omega$, and will occupy volume dV. The differentials transform according to $dk' = D^{-1}dk$, $d\Omega' = D^2 d\Omega$, and $dV' = DdV$, so that we find

$$dN = 2^{-2}\pi^{-3}[\exp(\hbar ck/k_B DT') - 1]^{-1} k^2 dk d\Omega\, dV.$$

All the factors of D cancel out except the one which multiplies T' in the Planck factor. Thus the only effect of the motion is to make the radiation temperature be dependent upon angle

$$T = DT' = T'\gamma^{-1}(1 - \beta\cos\theta)^{-1}$$
$$\approx T'(1 + \beta\cos\theta), \text{ for } \beta \ll 1.$$

This same result has been found by a different scheme for counting photons,[11] and by transforming the stress-energy-momentum tensor of the radiation field.[12] This anisotropy has indeed been observed.[13]

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