Relativistic transformation of solid angle

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We rederive the relativistic transformations of light intensity from compact sources to show where and how the transformation of solid angle contributes. We discuss astrophysical and other applications of the transformations.

INTRODUCTION

Two recent articles in this Journal have presented different aspects of the relativistic transformation of light intensity from compact sources (e.g., stars). In the first[1] the observer was considered to be moving, and in the second[2] the source was considered to be moving. Different expressions for the transformation were found in the two cases, which would seem to indicate either a mistake or an apparent violation of the principle of relativity. The difference lies in whether or not one takes into account the transformation of the solid angle associated with a collection of photons. In this investigation both expressions are rederived in yet another way to show that they are properly different in the way solid angle is treated. In this work the process of detection is emphasized, whereas in the earlier works attention was concentrated upon emission and propagation. We see that the intensity transformation for the special case of stars or galaxies receding along the line of sight has been known for decades. Other consequences of the solid-angle transformation are discussed for both moving sources and moving observers.

DERIVATION OF TRANSFORMATIONS

Let us consider the source to be fixed at the arbitrary point $\mathbf{r}$ in reference frame S. The source will be described in terms of the photons it emits. Each photon in turn is characterized by its propagation four vector $(k;\mathbf{k})$. Let us specify that during a time interval dt the source emits a number of photons with $\mathbf{k}$ in the increment $dk\,d\Omega$ given by $n(\mathbf{k})dk\,d\Omega\,dt$. As the photons propagate away, the same number crosses any fixed surface within the solid angle $d\Omega$ during an equal interval dt. Let us place a detector with directed aperture $\boldsymbol{\sigma}$ at the origin (Fig. 1). It presents normal area $\mathbf{i}\cdot\boldsymbol{\sigma}$ to the radiation and subtends solid angle $\mathbf{i}\cdot\boldsymbol{\sigma}/r^2$ as viewed from the source, where $\mathbf{i}$ is a unit vector directed from source to origin. Thus $\mathbf{r} = -r\mathbf{i}$, and $\mathbf{k} = k\mathbf{i}$ for the photons which are detected. The number of photons detected during dt is

$$dN_1 = n(\mathbf{k})dk\,dt\,\mathbf{i}\cdot\boldsymbol{\sigma}/r^2. \qquad (1)$$

Each photon has energy $\hbar ck$, so that the energy received during dt is

$$dE = \int_0^\infty \hbar ckn(\mathbf{k})dk\,dt\,\mathbf{i}\cdot\boldsymbol{\sigma}/r^2. \qquad (2)$$

This can be expressed as an intensity (energy per unit area per unit time)

$$I_0(r) = \frac{dE}{\mathbf{i}\cdot\boldsymbol{\sigma}\,dt} = \int_0^\infty \hbar ckn(\mathbf{k})dk/r^2. \qquad (3)$$

It can also be expressed in terms of a detected angular distribution (energy per unit solid angle per unit time)

$$Q(\mathbf{i}) = \frac{dE}{d\Omega dt} = \int_0^\infty \hbar ckn(\mathbf{k})dk. \qquad (4)$$

Now let us consider that a detector described by aperture $\mathbf{s}$ is moving at velocity $\mathbf{v} = \boldsymbol{\beta}c$, relative to frame S. This detector is considered to be fixed at the origin of another reference frame S', and the origins coincide at the instant of observation ($t = 0 = t'$). Thus the standard Lorentz transformation applies. During dt the moving detector sees a different number of photons than that given by Eq. (1) because it sweeps across other photons. The numerical density of photons in the vicinity of the origin is $n(\mathbf{k})dk/cr^2$. During time interval dt the detector sweeps out volume $dV = -\mathbf{s}\cdot\mathbf{v}\,dt$, where the volume is positive if the entrance window faces forward. The number of photons encountered by the moving detector during dt is

$$\begin{aligned} dN_2 &= n(\mathbf{k})dk\,dt\;\mathbf{i}\cdot\mathbf{s}/r^2 - \mathbf{s}\cdot\mathbf{v}\,dtn(\mathbf{k})dk/cr^2 \\ &= n(\mathbf{k})dk\,dt(\mathbf{i}-\boldsymbol{\beta})\cdot\mathbf{s}/r^2. \end{aligned} \qquad (5)$$

This expression can also be interpreted as the arrival at $\mathbf{s}$ of photons moving at the net velocity $c\mathbf{i} - \mathbf{v} = c(\mathbf{i} - \boldsymbol{\beta})$ as calculated in frame S.

Before we calculate the energy this represents, we must transform some of the quantities to frame S'. First there is the time dilatation observed for a clock attached to the detector

$$dt = (1-\beta^2)^{-1/2}dt' = \gamma\,dt'. \qquad (6)$$

Second there is the Lorentz–Fitzgerald contraction of the structure of the detector. Using $\perp$ and $\parallel$ to designate component vectors respectively perpendicular and parallel to $\mathbf{v}$, we have

$$\mathbf{s}_\parallel = \mathbf{s}'_\parallel, \quad \mathbf{s}_\perp = \mathbf{s}'_\perp/\gamma. \qquad (7)$$

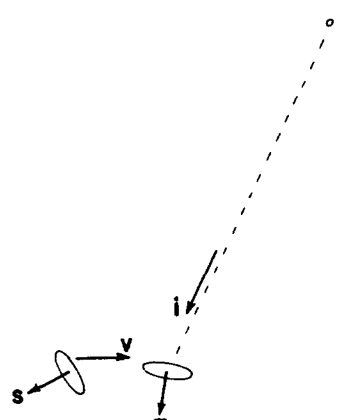

Fig. 1. Light from a distant source propagates in direction $\mathbf{i}$ and strikes a detector at the origin. The detector may be at rest relative to the source, in which case we describe it by directed aperture $\boldsymbol{\sigma}$. Alternatively, the detector may be moving past the origin at velocity $\mathbf{v}$, in which case we describe it in this reference frame by aperture $\mathbf{s}$.

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Third, there is the Lorentz transformation of the propagation four vector which leads to Doppler shift and aberration. We can write it as

$$k' = \gamma(k - \boldsymbol{\beta}\cdot\mathbf{k}) = \gamma(1-\boldsymbol{\beta}\cdot\mathbf{i})k = Dk, \quad (8)$$

$$\mathbf{k}'_{\parallel} = \gamma(\mathbf{k}_{\parallel} - \boldsymbol{\beta}k) \quad \text{or} \quad D\mathbf{i}'_{\parallel} = \gamma(\mathbf{i}_{\parallel} - \boldsymbol{\beta}), \quad (9)$$

$$\mathbf{k}'_{\perp} = \mathbf{k}_{\perp} \quad \text{or} \quad D\mathbf{i}'_{\perp} = \mathbf{i}_{\perp}. \quad (10)$$

The inverse transformations can be written as

$$k = \gamma(k' + \boldsymbol{\beta}\cdot\mathbf{k}') = \gamma(1+\boldsymbol{\beta}\cdot\mathbf{i}')k' = D^{-1}k', \quad (11)$$

$$\mathbf{k}_{\parallel} = \gamma(\mathbf{k}'_{\parallel} + \boldsymbol{\beta}k'), \quad \text{or} \quad D^{-1}\mathbf{i}_{\parallel} = \gamma(\mathbf{i}'_{\parallel} + \boldsymbol{\beta}). \quad (12)$$

The scalar product in Eq. (5) can now be transformed:

$$\begin{aligned}(\mathbf{i}-\boldsymbol{\beta})\cdot\mathbf{s} &= (\mathbf{i}_{\parallel}-\boldsymbol{\beta})\cdot\mathbf{s}_{\parallel} + \mathbf{i}_{\perp}\cdot\mathbf{s}_{\perp} \\ &= (D/\gamma)\mathbf{i}'_{\parallel}\cdot\mathbf{s}'_{\parallel} + (D\mathbf{i}'_{\perp})\cdot(\mathbf{s}'_{\perp}/\gamma) \\ &= (D/\gamma)\mathbf{i}'\cdot\mathbf{s}'. \quad (13)\end{aligned}$$

The number of photons encountered by the moving detector is an invariant and is written as

$$dN_2 = n(\mathbf{k})dk(\gamma\, dt')(D/\gamma)\mathbf{i}'\cdot\mathbf{s}'/r^2. \quad (5')$$

In frame S' each photon has energy $\hbar ck' = D\hbar ck$, so that the energy absorbed by the detector is measured in its rest frame as

$$dE' = \int_0^\infty D\hbar ckn(\mathbf{k})dk\, dt' D\mathbf{i}'\cdot\mathbf{s}'/r^2. \quad (14)$$

Using Eq. (3), we find the intensity in frame S' to be

$$I' = \frac{dE'}{\mathbf{i}'\cdot\mathbf{s}'dt'} = D^2 I_0(r). \quad (15)$$

Finally, let us transform the distance measurement into frame S'. Because the only connection between the source and the detector is the light signal between them, the physical significance of r is the spatial part of the displacement between the event points of emission and detection. That four-vector displacement is a null vector parallel to the propagation four vector. Thus, it satisfies transformation equations analogous to those of aberration and Doppler effect. The analog of Eq. (8) gives us at once

$$r' = Dr. \quad (16)$$

Upon substitution into Eq. (14) this yields

$$dE' = D^4 \int_0^\infty \hbar ckn(k)dk\, dt'\mathbf{i}'\cdot\mathbf{s}'/r'^2. \quad (14')$$

This can be interpreted in two ways. In terms of intensity

$$I' = \frac{dE'}{\mathbf{i}'\cdot\mathbf{s}'dt'} = D^4 I_0(r'). \quad (17)$$

With the identification $d\Omega' = \mathbf{i}'\cdot\mathbf{s}'/r'^2$ for the solid angle subtended by the detector as measured in frame S', this yields a transformation of detected angular distribution

$$Q'(\mathbf{i}') = \frac{dE'}{d\Omega' dt'} = D^4 Q(\mathbf{i}). \quad (18)$$

DISCUSSION

Here a consistent derivation has led to two expressions [Eqs. (15) and (17)] which both relate intensity measurements in two reference frames, but which have different powers of the Doppler factor D. The first transformation [Eq. (15)] applies when two detectors occupy different reference frames, but measure equivalent collections of photons at the same event point. This transformation $I' = D^2 I$ specifically applies for plane waves and was derived in Ref. 1 by a different method of counting photons. One factor D is attributed to the transformation of the energy of each photon, and the other factor D is attributed to the transformation of the spacing of photons along their direction of propagation. This transformation can also be found by other methods described further in Ref. 1. All of these methods have in common that they depend only upon the transformation properties of propagating radiation, and do not depend upon the nature or motion of the source. Thus, this transformation can be considered as the most fundamental one. It continues to apply for divergent light, as here and in Ref. 1. The only effect of divergence is to add into Eq. (5) terms of higher order in the small quantities dt and $\mathbf{s}$.

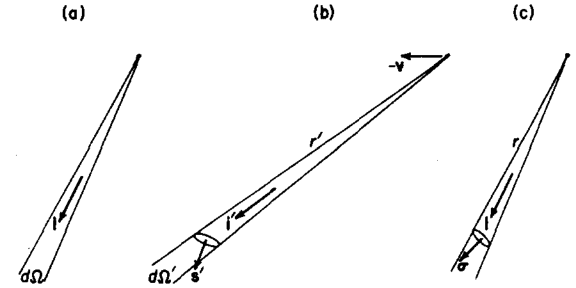

Fig. 2. (a) Viewed in its rest frame S, the source emits light into $d\Omega$ along $\mathbf{i}$. (b) Viewed in frame S', in which the source is moving at velocity $-\mathbf{v}$, the same light occupies $d\Omega' = D^{-2}d\Omega$ along $\mathbf{i}'$. The solid angle $d\Omega'$ is defined by the detector $\mathbf{s}'$ with normal area $\mathbf{i}'\cdot\mathbf{s}'$. (c) Viewed in frame S, a detector $\boldsymbol{\sigma}$ presents the same normal area $\mathbf{i}\cdot\boldsymbol{\sigma} = \mathbf{i}'\cdot\mathbf{s}'$. The solid angle $d\Omega$ is the same as in (a), but the separation is $r = D^{-1}r'$.

In the second transformation [Eq. (17)] the source is considered to be moving. The additional factor D^2 found here from the transformation of apparent distance can also be associated with solid angle as follows. For any beam of photons, the area they occupy normal to their direction of propagation is invariant, as shown by Weisskopf[3] and in Ref. 1. This means that $\mathbf{i}\cdot\boldsymbol{\sigma}$ in Eq. (2) and $\mathbf{i}'\cdot\mathbf{s}'$ in Eq. (14) can be taken equal. From Eq. (16) and that equality we have

$$d\Omega' = \mathbf{i}'\cdot\mathbf{s}'/r'^2 = \mathbf{i}\cdot\boldsymbol{\sigma}/(Dr)^2 = D^{-2}d\Omega. \quad (19)$$

This can also be found by differentiating the component form of Eq. (9) for aberration[2,3]:

$$\cos\theta' = (\cos\theta - \beta)/(1-\beta\cos\theta); \quad (9')$$

$$\frac{d\Omega'}{d\Omega} = \frac{d(\cos\theta')d\phi'}{d(\cos\theta)d\phi} = \frac{1-\beta^2}{(1-\beta\cos\theta)^2} = D^{-2}. \quad (19')$$

The significance of this transformation of solid angle can be found by working backward through the intensity transformations. The source in frame S is moving at velocity $-\mathbf{v}$ relative to frame S'. The light which it emits into solid angle $d\Omega$ at direction $\mathbf{i}$ in its rest frame [Fig. 2(a)] appears in solid angle $d\Omega' = D^{-2}d\Omega$ in frame S' [Fig. 2(b)]. The transformation of detected angular distribution [Eq. (18)] thus includes four factors of D attributed as follows[2]: one factor from the energy of each photon, one factor from the spacing of photons along their direction of propagation, and two factors from the change of solid angle. The solid angle $d\Omega'$ can be defined by the detector s′ at distance r', all measured in S'. The intensity transformation [Eq. (17)] compares the energy absorbed by s′ to that which would be produced by an identical source at rest in S' at the same distance r'. Note that if the source is not spherically symmetric, then the comparison source must be rotated through the angle between **i** and **i**′. This is exactly the aberration-rotation angle involved in the visual appearance of moving objects.[3] It vanishes for motion along the line of sight, i.e., for **v** parallel or antiparallel to **i**. Any possible angular dependence on **i** was suppressed in Eq. (17).

At the same event point where the light from the actual source strikes s′, we now consider another detector σ in S. It is chosen so that it intercepts exactly the same cross-sectional area of light in S as s′ does in S'. The solid-angle transformation means that in frame S, in which both the source and this detector are fixed, the intercepted light has the same solid angle $d\Omega$ we started with. This in turn requires that the separation of source and detector be $r = r'/D$ [Fig. 2(c)]. The intensity measured by this detector has the obvious value given by the fundamental intensity transformation and Eqs. (16) and (17):

$$I = D^{-2}I' = D^{-2}[D^4 I_0(r')] = D^2 I_0(Dr) = I_0(r).$$

In a private communication, B. W. Augenstein has pointed out two references which present special cases of Eq. (17). [Because that equation is so closely related to Eq. (18), this effectively refutes the statement in Ref. 2 that this transformation had never been presented.] McCrea[4] derived Eq. (17) for the special case of a star receding along the line of sight, in which case $D = [(1-\beta)/(1+\beta)]^{1/2}$. In a study of cosmology without general relativity,[5] Milne considered an extended source ("nebula" or galaxy) and carried the derivation to about the stage of Eq. (5) for arbitrary angle between **i** and **v**. However, he specialized to recession along the line of sight before completing the transformation to the frame of the detector. Note that the general expression found here is the simplest possible extension from the special case of direct recession. One need only measure D from a study of lines in the spectrum. It is immaterial what combination[6] of radial motion and astronomical proper motion gives rise to D—the intensity from the moving source is D^4 times the intensity from an equivalent source at rest at that apparent distance. If the source is a blackbody at temperature T in its own frame, it will appear to have a Doppler-shifted temperature DT when moving.[1] The change in intensity then follows directly from the Stefan–Boltzmann radiation law.

The transformation of solid angle has a mixed history in other investigations. It was actually the only factor in the transformation of intensity which was included in one presentation of the visual appearance of rapidly moving objects.[3] It was properly included (attributed to Ref. 4) in an explanation of the dark night sky paradox by means of the expansion of the universe.[7] It was originally omitted in the first derivation[8] of the expected anisotropy of the 3 K cosmic blackbody radiation due to earth's motion. It has been included in subsequent derivations.[2,9] This last astrophysical application is distinct from all the other applications mentioned here in that the source is inescapably diffuse rather than compact and the solid angle of acceptance is an intrinsic feature of the detector and of cavity radiation itself.

The transformation of solid angle would also have a subtle effect on the appearance of the starfield from a moving spaceship, which was not mentioned in Ref. 1. Any resolvable stellar disk would appear to have its subtended solid angle multiplied by D^{-2}, or equivalently its angular diameter multiplied by D^{-1}. (It always presents a circular outline.[10]) This is entirely consistent with the transformation of its apparent distance. A stereoscopic range finder on the ship would also show each star at the transformed distance $r' = Dr$.[11] Thus one has the amusing circumstance that the forward stars ($D > 1$) would all appear to get further away as the ship's speed increases. The various sizes of disks in the star plots of Ref. 1 describe only the apparent brightness of each star, and not its resolved outline.

It is hoped that this presentation can clarify the role of the solid-angle transformation. Because it is a necessary consequence of aberration, there is always a change in the solid angle associated with light when one changes reference frame (except for the case of idealized plane waves). Its effect upon the transformation of intensity must be included in the study of the cosmic blackbody radiation, where the angle is associated with convergence into the detector of radiation from different parts of the primordial fireball. It must also be included when a moving source is compared to an equivalent fixed source, both at the same apparent distance measured in the reference frame of the detector. In this case the angle represents divergence from the compact source to the detector. The change in solid angle does not contribute to the transformation of intensity measured at the same event point by detectors in relative motion. Note particularly that the amount of divergence of the light, or equivalently the curvature of its wave fronts, does not determine which intensity transformation is correct. Thus, in considering the appearance of stars from a moving spaceship, a typical source distance is 1–1000 pc but the intensity transformation is equivalent to that for plane waves. On the other hand, typical receding galaxies are 10^6 times further away (so that their light is actually more nearly plane), but their light cannot be treated as plane waves in finding its intensity transformation.

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