Differential aging from acceleration: An explicit formula

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We consider a clock paradox where an observer leaves an inertial frame, is accelerated, and after an arbitrary trip returns. We discuss a simple equation that gives an explicit relation in $1+1$ dimensions between the time elapsed in the inertial frame and the acceleration measured by the accelerating observer during the trip. A non-closed trip with respect to an inertial frame appears closed with respect to another suitable inertial frame. We use this observation to define the differential aging as a function of proper time. The reconstruction problem of special relativity is discussed and it is shown that its solution would allow the construction of an inertial clock. © *2005 American Association of Physics Teachers.*

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I. INTRODUCTION

The differential aging implied by special relativity is an astonishing result.[1–4] The relation between the role of acceleration and the difference in proper times of inertial and accelerated observers has been widely discussed.[5–17] Acceleration should not be regarded as the source of differential aging, but the opposite point of view has sometimes been considered because an observer can undergo a round-trip journey only by accelerating in Minkowski space–time. However, examples of differential aging can easily be given for which no acceleration is required. These examples can be found in curved space–times or in space–times with nontrivial topology.[18–26] These considerations do not imply that differential aging cannot be expressed in Minkowski space–time as a function of the acceleration. We address this last problem in this paper.

II. THE TEXTBOOK ROUND-TRIP EXAMPLES

Here we give a relation, Eq. (1), which relates acceleration to differential aging when the accelerated observer undergoes an unidirectional, but otherwise arbitrary, motion. We shall derive Eq. (1) in Sec. III. First, we discuss and apply Eq. (1) to some cases previously investigated with methods especially adapted to the particular cases considered. Those methods, while often elementary, cannot be used to study the general case that is included in our analysis.

We choose units such that $c=1$. Let K be an inertial frame and choose coordinates in a way such that two of them can be suppressed. Let O be an accelerated observer with timelike worldline x^{μ} with $x^1(0)=x^1(\bar{\tau})=0$, where τ is the proper time, $\bar{\tau}$ is the proper time according to the accelerated observer at the end of the trip, and x^{μ} ($\mu=0,1$) are the coordinates of the inertial frame. Let $a(\tau)$ be the acceleration of O with respect to the local inertial frame at $x(\tau)$. More precisely, the quantity $-a$ is the apparent acceleration measured by O, and hence it has a positive or negative sign depending on the direction. Let $T=x^0(\bar{\tau})-x^0(0)$ be the (positive) inertial time interval between the departure and arrival of O. The time dilation T is related to the acceleration $a(\tau)$ by

$$T^2=\left[\int_0^{\bar{\tau}} e^{\int_0^{\tau} a(\tau')d\tau'} d\tau\right]\left[\int_0^{\bar{\tau}} e^{-\int_0^{\tau} a(\tau')d\tau'} d\tau\right]. \tag{1}$$

Equation (1) is the time dilation-acceleration equation. Also the accelerated observer departs from K with zero velocity if and only if $\int_0^{\bar{\tau}} e^{\int_0^{\tau} a(\tau')d\tau'} d\tau=\int_0^{\bar{\tau}} e^{-\int_0^{\tau} a(\tau')d\tau'} d\tau'$. In this case,

$$T=\int_0^{\bar{\tau}} e^{\pm\int_0^{\tau} a(\tau')d\tau'} d\tau. \tag{2}$$

If, moreover, the final velocity of O with respect to K vanishes, then $\int_0^{\bar{\tau}} a(\tau)d\tau=0$.

Some comments are in order. In Eqs. (1) and (2) the initial and final velocity of O with respect to K does not appear. If we use the Cauchy–Schwarz inequality $(\int fg\,d\tau)^2 \leqslant (\int f^2 d\tau)(\int g^2 d\tau)$, with $f=g^{-1}=\exp(\int_0^{\tau} a(\tau')d\tau'/2)$, we find the expected relation $T\geqslant\bar{\tau}$, where the equality holds only if $f=kg$, for a particular constant k, that is, if and only if $a(\tau)=0$. Thus either $T>\bar{\tau}$ or the worldline of O coincides with that of the origin of K. We have hence obtained the differential aging effect. In Sec. IV we shall give another proof that does not use the Cauchy–Schwarz inequality.

Often the differential aging effect is derived in curved (and hence even in flat) space–times by noticing that the connecting geodesic, that is, the trajectory of the equation $x^1(\tau)=0$ locally maximizes the proper time functional $I[\gamma]=\int_{\gamma} d\tau$.[27] Equation (1) implies the global maximization property in $1+1$ Minkowski space–time and has the advantage of giving an explicit result for the inertial round-trip dilation.

A. The simplest example

The simplest example is that of uniform motion in two intervals $[0,\bar{\tau}/2]$ and $[\bar{\tau}/2,\bar{\tau}]$. In the first interval O moves with respect to K with velocity $v=dx^1/dx^0$ and in the second interval with velocity $-v$. Although this example is elementary, it is interesting to see how Eq. (1) predicts the usual result $T=\bar{\tau}/\sqrt{1-v^2}$. The first problem is that Eq. (1) holds for integrable acceleration functions. In this example the acceleration has a singularity at $\bar{\tau}/2$. The initial and final singularities are not present if the motion of O is not forced to coincide with that of K's origin for τ outside the interval.

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The reader can easily check (or see Sec. III) that if $\theta(\tau) = \tanh^{-1} v(\tau)$ is the rapidity, then $d\theta/d\tau = a$. (This result follows from the additivity of the rapidity under boosts and the fact that a small increment in rapidity coincides with a small increment in velocity with respect to the local inertial frame.) Hence

$$\Delta\theta = \int a\, d\tau. \tag{3}$$

If the acceleration causes, in an arbitrary small interval centered at $\tilde{\tau}$, a variation $\Delta\theta$ in the rapidity, then we must include a term $\Delta\theta\delta(\tau - \tilde{\tau})$ in the expression for $a(\tau)$. With this rule, Eq. (1) also holds for accelerations $a(\tau)$ with Dirac's delta singularities. However, in this case it is no longer true that T does not depend on the initial and final velocities of O, and we need to use this information to find the coefficient $\Delta\theta$. In our example we have

$$\Delta\theta = \tanh^{-1}(-v) - \tanh^{-1} v = -2 \tanh^{-1} v. \tag{4}$$

We substitute $a = -2 \tanh^{-1} v\, \delta(\tau - \tilde{\tau})$ in Eq. (1) and find, after some manipulations with hyperbolic functions, that $T = \bar{\tau}/\sqrt{1 - v^2}$ as expected.

We should not be surprised that this simple case needs so much work, because it is a pathological case. No real observer would survive an infinite acceleration. The advantage of Eq. (1) becomes evident in more realistic cases.

B. Constant acceleration

This case also has been treated extensively.[28,29] The assumption is that in the interval $[0, \bar{\tau}]$ we have $a = g$ with $g \in \mathbb{R}$. Equation (1) gives

$$T^2 = \left[\int_0^{\bar{\tau}} e^{g\tau} d\tau\right]\left[\int_0^{\bar{\tau}} e^{-g\tau} d\tau\right] = \frac{2}{g^2}(\cosh g\bar{\tau} - 1), \tag{5}$$

or $T = (2/g)\sinh(g\bar{\tau}/2)$.[30,31]

C. A more complicated example

This example was considered by Taylor and Wheeler.[32,33] Its advantage is that the acceleration has no Dirac's delta functions, and O departs from and arrives at K with zero velocity. The interval is divided into four equal parts of proper time duration $\bar{\tau}/4$. The acceleration in these intervals is successively g, $-g$, $-g$, and g (Fig. 1).

We can easily convince ourselves that because the acceleration in the second interval is opposite to the one in the first interval, the observer returns to K's worldline. Moreover, we know that O starts with zero velocity so we can apply Eq. (2). First we have

$$\int_0^{\tau} a(\tau')d\tau' = \begin{cases} g\tau, & \tau \in [0, \frac{1}{4}\bar{\tau}] \\ -g\tau + g\bar{\tau}/2, & \tau \in [\frac{1}{4}\bar{\tau}, \frac{3}{4}\bar{\tau}] \\ g\tau - g\bar{\tau}, & \tau \in [\frac{3}{4}\bar{\tau}, \bar{\tau}]. \end{cases} \tag{6}$$

We then have

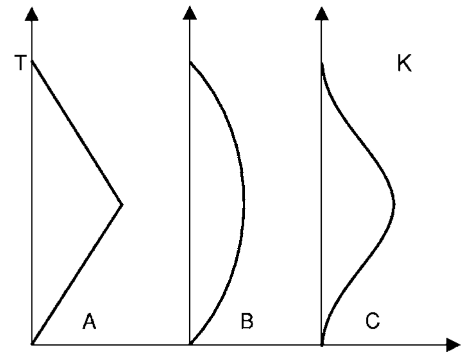

Fig. 1. The textbook round-trip examples.

$$\begin{aligned} T^2 = &\left[\int_0^{(1/4)\bar{\tau}} e^{g\tau} d\tau + e^{g\bar{\tau}/2}\int_{(1/4)\bar{\tau}}^{(3/4)\bar{\tau}} e^{-g\tau} d\tau \right. \\ &\left. + e^{-g\bar{\tau}}\int_{(3/4)\bar{\tau}}^{\bar{\tau}} e^{g\tau} d\tau\right]\left[\int_0^{(1/4)\bar{\tau}} e^{-g\tau} d\tau \right. \\ &\left. + e^{-g\bar{\tau}/2}\int_{(1/4)\bar{\tau}}^{(3/4)\bar{\tau}} e^{g\tau} d\tau + e^{g\bar{\tau}}\int_{(3/4)\bar{\tau}}^{\bar{\tau}} e^{-g\tau} d\tau\right] \\ = &\frac{1}{g^2}[(e^{g\bar{\tau}/4} - 1) + (e^{g\bar{\tau}/4} - e^{-g\bar{\tau}/4}) + (1 - e^{-g\bar{\tau}/4})] \\ &\times [(1 - e^{-g\bar{\tau}/4}) + (e^{g\bar{\tau}/4} - e^{-g\bar{\tau}/4}) + (e^{g\bar{\tau}/4} - 1)], \end{aligned} \tag{7}$$

and obtain $T = (4/g)\sinh(g\bar{\tau}/4)$.[33]

III. THE RECONSTRUCTION PROBLEM IN SPECIAL RELATIVITY

We consider the problem of reconstructing the motion in the inertial frame starting from a knowledge of the acceleration. Similar mechanical problems have been considered in Refs. 34–36. We recall that in n-dimensional Minkowski space–time (we use the timelike convention $\eta_{00} = 1$), as well as in the curved space–time of general relativity,[37] an observer is represented by a timelike worldline $x(\tau)$ and by an assignment at each point of $x(\tau)$ of n orthonormal vectors $\{u(\tau), e_1(\tau), \ldots, e_{n-1}(\tau)\}$ (the normalization depends on the metric signature, in our case $u \cdot u = 1$ and $e_i \cdot e_i = -1$, $i = 1, 2 \ldots n-1$), where $u = \partial_\tau$ is the covariant velocity. The vectors e_i determine the orientation of the observer as she moves in space–time. In the two-dimensional case that interests us, the orientation of e_1 is uniquely determined by the condition of orthogonality with u. The vectors e_i are Fermi–Walker-transported[37] if $\nabla_u e_i^\mu = (a^\mu u_\nu - u^\mu a_\nu)e_i^\nu$ $(\mu, \nu = 0, 1, \ldots, n-1)$. In this case the observer preserves her orientation with respect to $n-1$ orthogonal comoving gyroscopes which can be identified with the vectors e_i. The special form of the Fermi–Walker transport arises from the requirement of orthonormality of the set $\{u(\tau), e_1(\tau), \ldots, e_{n-1}(\tau)\}$, which should be preserved along the observer worldline. Notice that if the observer acceleration vanishes, then the Fermi–Walker transport coincides with the parallel transport.

The reconstruction problem of special relativity can be conveniently stated in n-dimensional Minkowski space–time as follows. Consider a timelike worldline $x^\mu(\tau)$ in

Minkowski space–time and $n-1$ orthonormal Fermi–Walker-transported vectors e_i representing the comoving gyroscopes. Let $a^i(\tau)=-a(\tau)\cdot e_i$ be the components of the acceleration vector $a=a^i e_i$ with respect to the gyroscope directions. Determine, starting from the data $a^i(\tau)$, the original curve up to an affine transformation of Minkowski space–time.

Note that the components of the acceleration with respect to the gyroscope directions are measurable by O using $n-1$ orthogonal gyroscopes and an accelerometer. The solution to this problem for $n=4$ may be relevant for future space travelers. Although the twin paradox usually has been studied assuming the possibility of communication by light signals, it is more likely that when distances increase, communication would become impossible. Suppose the space traveler does not want to be lost, but still wishes to choose her own trajectory. In this case she needs to find some way to know her inertial coordinates. The only way, if no references in space are given, is to solve the reconstruction problem. By keeping track of her acceleration during the journey, she would be able to reconstruct her inertial coordinates without looking outside the laboratory. In particular, she would be able to construct (merging an accelerometer, three gyroscopes, and an ordinary clock) an *inertial clock*, that is, a clock that displays $x^0(\tau)$.

The solution to the reconstruction problem also gives to O the advantage of knowing her own position even before K knows it. Indeed, O can know $x^\mu(\tau)$ ($\mu=0,1,2,3$) immediately, while K has to wait for light signals from O. To perturb her trajectory, O can immediately apply corrections, while for great distances, a decision from K would take too much time.

The $1+1$ *dimensional case*. In $1+1$ dimensions the reconstruction problem can be solved easily. In particular, the vector $e_1(\tau)$ is automatically Fermi–Walker transported due to the orthogonality condition with u. For higher dimensions the problem becomes much more complicated and numerical methods need to be used. We give the solution to the $1+1$ case. Greek indices take the values 0 and 1.

If $v^\mu=dx^\mu/dx^0$, $v=dx^1/dx^0$, and $u^\mu=dx^\mu/d\tau$, we have

$$a^\mu=\frac{du^\mu}{d\tau}=\frac{d}{d\tau}\frac{v^\mu}{\sqrt{1-v^2}}. \tag{8}$$

Let $(0,a)$ be the components of the acceleration in the local inertial frame [the first component vanishes because $u\cdot a=(1/2)(d/d\tau)(u\cdot u)=0$]. Because the square of the acceleration is a Lorentz invariant, we have $-a^2=a^\mu a_\mu$, or

$$-a^2=\left(\frac{d}{d\tau}\frac{1}{\sqrt{1-v^2}}\right)^2-\left(\frac{d}{d\tau}\frac{v}{\sqrt{1-v^2}}\right)^2 \tag{9a}$$

$$=-\frac{1}{(1-v^2)^2}\left(\frac{dv}{d\tau}\right)^2. \tag{9b}$$

But a has the same sign as $dv/d\tau$ and hence $a=d\theta/\tau$, where $\theta=\tanh^{-1}v$ is the rapidity. We have

$$v(\tau)=\tanh\left[\int_0^\tau a(\tau')d\tau'+\tanh^{-1}v(0)\right]. \tag{10}$$

From $dx^0=d\tau/\sqrt{1-v^2}$ and $dx^1=d\tau v/\sqrt{1-v^2}$, we obtain

$$x^0(\tau)-x^0(0)=\int_0^\tau\cosh\left[\int_0^{\tau'}a(\tau'')d\tau''+\tanh^{-1}v(0)\right]d\tau', \tag{11a}$$

$$x^1(\tau)-x^1(0)=\int_0^\tau\sinh\left[\int_0^{\tau'}a(\tau'')d\tau''+\tanh^{-1}v(0)\right]d\tau'. \tag{11b}$$

Note that $v(0)$ also is easily measurable by O because at $\tau=0$, K and O cross each other. Without knowing $v(0)$, the inertial coordinates are determined only up to a global affine transformation. We may say that a knowledge of $v(0)$ specifies, up to translations, the inertial coordinates and inertial frame with respect to which we describe O's motion.

Now consider the invariant under affine transformations

$$T^2(\tau)=[x^0(\tau)-x^0(0)]^2-[x^1(\tau)-x^1(0)]^2. \tag{12}$$

Because $x(\tau)$ is in the chronological future of $x(0)$, there is a timelike geodesic passing through the two events. The motion of the accelerated observer is a round-trip with respect to the inertial observer $K(\tau)$ moving along that geodesic. Let $x^\mu_{K(\tau)}$ be the coordinates of O with respect to $K(\tau)$. We have $x^1_{K(\tau)}(\tau)=x^1_{K(\tau)}(0)=0$, and thus the invariant T reads

$$T(\tau)=x^0_{K(\tau)}(\tau)-x^0_{K(\tau)}(0). \tag{13}$$

That is, $T(\tau)$ is the travel duration with respect to an inertial observer that sees the motion of the accelerated observer as a round trip that ends at τ. By using the relation $a^2-b^2=(a-b)(a+b)$, we have from Eq. (12)

$$T^2(\tau)=\left[\int_0^\tau e^{\int_0^{\tau'}a(\tau'')d\tau''}d\tau'\right]\left[\int_0^\tau e^{-\int_0^{\tau'}a(\tau'')d\tau''}d\tau'\right]. \tag{14}$$

Remarkably the dependence on $v(0)$ disappears, which follows from the fact that contrary to $x^0(\tau)$ and $x^1(\tau)$, the quantity $T(\tau)$ is a Lorentz invariant and hence should not depend on the choice of inertial frame, that is, the choice of $v(0)$.

To derive Eq. (2), note that if O departs with zero velocity, then from Eq. (11b), after imposing the round-trip condition $x^1(\bar{\tau})=x^1(0)$, it follows that

$$\int_0^{\bar{\tau}}\sinh\left[\int_0^\tau a(\tau')d\tau'\right]d\tau=0. \tag{15}$$

That is, the two factors in Eq. (1) for T^2 coincide. Finally, if O departs and returns with zero velocity, we have $\int_0^{\bar{\tau}}a\,d\tau=0$, which follows from the already derived relation $a=d\theta/d\tau$.

IV. DIFFERENTIAL AGING

We now give another derivation that $T(\bar{\tau})>\bar{\tau}$, unless $a(\tau)=0$ for all $\tau\in[0,\bar{\tau}]$, in which case $T(\bar{\tau})=\bar{\tau}$ and O is at rest in K. The idea is to define differential aging, even for proper times $\tau<\bar{\tau}$, as the differential aging between $K(\tau)$ and O. The differential aging at τ is therefore by definition $\Delta(\tau)=T(\tau)-\tau$, that is, the difference between the proper time elapsed for an inertial observer who reaches $x(\tau)$ from $x(0)$ and that elapsed in the accelerating frame. Roughly speaking, if at proper time τ the accelerating observer asks "What is the differential aging now?," the answer using this definition would be it is the differential aging between her and an imaginary twin sister who reached the same event where she is now, but moving along a geodesic. This definition has the advantage of avoiding conventions for distant simultaneity. Indeed, a distant simultaneity convention seems to be needed to give a meaning to the word *now* used in the previous question. However, the strategy of comparing directly the age of O with that of K, exploited in previous work, does not provide a unique solution because the accelerating observer can set up different coordinates (most studied are the radar,[38,39] Fermi,[37] and Møller[40] coordinates), and hence the observer can choose different simultaneity slices with each one leading to a different result.[38] Here, it is convenient to adopt the alternative strategy of comparing the age of O at her proper time τ not with $K=K(\bar{\tau})$, but with $K(\tau)$, $\tau<\bar{\tau}$, to study how the differential aging changes with τ and then, in the end, let $\tau=\bar{\tau}$.

The differential aging $\Delta(\tau)$ is a nondecreasing function

$$\frac{d\Delta}{d\tau}\geqslant 0, \tag{16}$$

where the equality holds for all $\tau'\in[0,\tau]$ iff $a(\tau')=0$ for all $\tau'\in[0,\tau]$. The derivation of Eq. (16) goes as follows. Let $\Theta(\tau)=\int_0^\tau a(\tau)d\tau$. The derivative of $T(\tau)$ is

$$\frac{dT}{d\tau}=\cosh A(\tau), \tag{17}$$

where

$$A(\tau)=\Theta(\tau)+\frac{1}{2}\ln\left[\frac{\int_0^\tau e^{-\Theta(\tau')}d\tau'}{\int_0^\tau e^{\Theta(\tau')}d\tau'}\right]. \tag{18}$$

Because $\cosh A\geqslant 1$, Eq. (16) follows.

Now, suppose $d\Delta(\tau)/d\tau=0$, then $A(\tau)=0$, or

$$e^{-2\Theta}=\frac{\int_0^\tau e^{-\Theta(\tau')}d\tau'}{\int_0^\tau e^{\Theta(\tau')}d\tau'}. \tag{19}$$

Assume $d\Delta(\tau')/d\tau=0$ for all $\tau'\in[0,\tau]$. Then Eq. (19) holds for all $\tau'\in[0,\tau]$. We differentiate Eq. (19) and obtain

$$-2a(\tau)e^{-2\Theta}=\frac{e^{-\Theta}\int_0^\tau e^{\Theta(\tau')}d\tau'-e^{\Theta}\int_0^\tau e^{-\Theta(\tau')}d\tau'}{(\int_0^\tau e^{\Theta(\tau')}d\tau')^2}=0, \tag{20}$$

that is, $a(\tau)=0$ for all $\tau'\in[0,\tau]$.

Because $\Delta(0)=0$, Eq. (16) implies that $\Delta(\tau)>0$ for $\tau>0$ unless $a(\tau')=0$ for all $\tau'\leqslant\tau$. This argument confirms again the differential aging effect. However, Eq. (16) also implies that the definition of differential aging is particularly well behaved, because as proper time passes, the imaginary twin gets older with respect to the accelerating observer.

V. CONCLUSIONS

We have discussed the reconstruction problem in special relativity and showed its relevance for the construction of inertial clocks and in general for the positioning of the space traveler. We gave a simple equation that relates the round-trip inertial time dilation with the acceleration measured by the noninertial observer and applied it to some well-known cases to show how it works in the presence of singularities. We believe that this equation could be useful for explaining the relation between the acceleration and the differential aging $T(\tau)-\tau$. Indeed, the differential aging effect is obtained easily by applying the Cauchy–Schwarz inequality.

Although the twin paradox is discussed in almost every textbook on special relativity, the discussion of examples with singularities is not always completely satisfactory and more refined examples require a lot of work. In contrast, the derivation of Eq. (1) is elementary and needs only some concepts from calculus. Its derivation would probably convince students of the reality of the differential aging effect.

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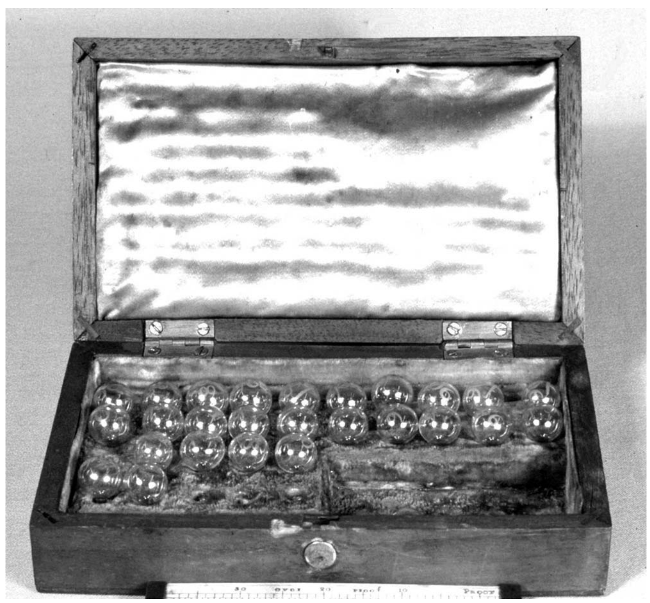

Specific Gravity Spheres. Middlebury College has an unusual set of glass specific gravity globes patented and [presumably] made by Lovi of Edinburgh. The globes are weighted to give them an increasing series of densities. It is sink or swim; the globes are placed one by one in the liquid to determine which one has the density closest to the liquid, at which point it has neutral buoyancy. The apparatus was probably designed for use in the distilling industry. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)