$$y_r = 2\cos(\chi 2^r)$$

as in Eq. (9).

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Wave front relativity

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A complete picture of the relativity of wave fronts, heretofore lacking, is presented in the context of an expanding spherical light wave as recorded in two Lorentz frames in relative motion.

I. INTRODUCTION

In this article I reconsider the relativity of wave fronts and give a detailed description, heretofore lacking, of how wave front events in one Lorentz frame form a wave front in a second in relative motion. I choose as a model the situation which Einstein introduced in his famous 1905 paper:[1] given two inertial frames of reference S and $\bar{S}$ with parallel axes and $\bar{S}$ moving at constant velocity $v=\beta c$ with respect to S in the direction of the positive x^3 axis; at the time $t=\bar{t}=0$, when the origins of the two frames coincide, a light pulse is emitted at the point of coincidence. The spherical wave front propagates in S radially outward from the origin O at the speed of light c and the events $(x)\equiv(x^0=ct, x^1, x^2, x^3)$ on this wave front satisfy

$$(x^1)^2+(x^2)^2+(x^3)^2=(x^0)^2. \tag{1}$$

If this equation is transformed to the $\bar{S}$ frame with the Lorentz transformation, $x^\mu=\Lambda^\mu{}_\nu \bar{x}^\nu$, one obtains

$$(\bar{x}^1)^2+(\bar{x}^2)^2+(\bar{x}^3)^2=(\bar{x}^0)^2, \tag{2}$$

which has been interpreted as describing a spherical wave front in $\bar{S}$ of radius $\bar{x}^0=c\bar{t}$ at time $\bar{t}$, centered on the origin $\bar{O}$. In fact, such an interpretation dates back to Einstein's original paper:[1] "The wave under consideration is therefore no less a spherical wave with velocity of propagation c when viewed in the moving frame." More recently French[2] has given a similar interpretation. Other authors[3–6] have used this model to derive the Lorentz transformation. While this widely accepted view seems compelling, I nevertheless contend that it obscures the complete picture of the relativity of wave fronts, which is more subtle and less mysterious than the Lorentz transformation from Eq. (1) to Eq. (2) above seems to imply.

In support of this contention I establish two principal results: (1) when the events on the spherical wave front in S at some arbitrary time t are labeled by a set of coordinates which are comoving[7] on the light cone, the Lorentz transformation unambiguously maps them in $\bar{S}$ onto an ellipsoid of revolution centered on O at time $\bar{t}=\gamma t\equiv t/\sqrt{1-\beta^2}$ and not a sphere centered on $\bar{O}$; (2) there exists a corresponding ellipsoid of events in S that make up a spherical wave front in $\bar{S}$; these events occur on a sequence of minor circles on the S wave front as it expands radially outward. The second result provides a complete picture of wave front relativity, including a detailed explanation of how the spherical wave fronts in the two frames can be centered on the respective origins, even though O and $\bar{O}$ are in relative motion.

II. THE ELLIPSOID OF S WAVE FRONT EVENTS IN THE $\bar{S}$ FRAME

The conceptual difficulty with the interpretation of Eq. (2) stems from the fact that the three spatial coordinates x^1, x^2, x^3 are not comoving with the points on the spherical wave front in S. Let us replace them with the two spherical angles, θ and ϕ, and denote a wave front event in S by (x^0,θ,ϕ). The spherical coordinates are comoving in the sense that a given point on the S wave front always has the same values of θ and ϕ. The rectangular spatial coordinates of an S wave front event are given by the usual equations,

$$x^1=x^0\sin\theta\cos\phi,\quad x^2=x^0\sin\theta\sin\phi,\quad x^3=x^0\cos\theta. \tag{3}$$

The Lorentz transformation to the barred frame is

$$\bar{x}^0=\gamma(x^0-\beta x^3), \tag{4}$$

$$\bar{x}^1=x^1,\quad \bar{x}^2=x^2, \tag{5}$$

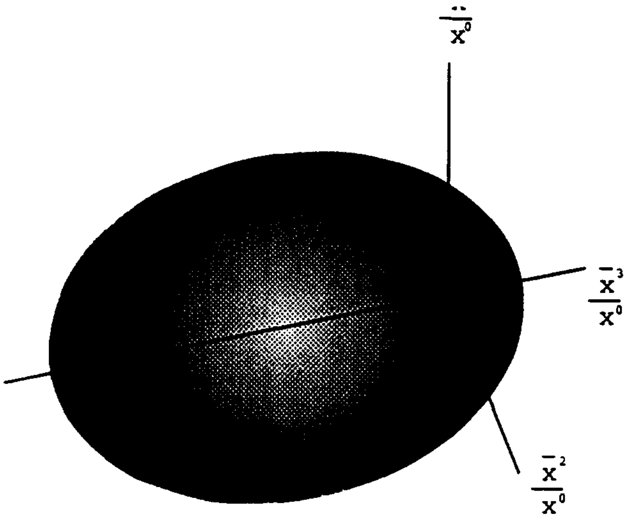

Fig. 1. The ellipsoid of events in the $\bar{S}$ frame that form a wave front in S at an arbitrary time t for $\beta = 1/\sqrt{2}$. The events occur sequentially in time from right to left and so do not constitute a wave front in $\bar{S}$.

$$\bar{x}^3 = \gamma(x^3 - \beta x^0). \tag{6}$$

Substituting Eqs. (3) into Eqs. (4)–(6) gives us the $\bar{S}$ coordinates of the set of wave front events (x^0, θ, ϕ) that occur simultaneously in S at some arbitrary time t and radius $x^0 = ct$.

$$\bar{x}^0 = x^0\gamma(1 - \beta\cos\theta), \tag{7}$$

$$\bar{x}^1 = x^0 \sin\theta\cos\phi, \quad \bar{x}^2 = x^0\sin\theta\sin\phi, \tag{8}$$

$$\bar{x}^3 = x^0\gamma(\cos\theta - \beta). \tag{9}$$

Equation (7) indicates that simultaneous wave front events in S with different values of θ occur at different times in $\bar{S}$ and so do not constitute a wave front in the barred frame. Furthermore, from Eqs. (8) and (9) one can easily show that

$$\frac{(\bar{x}^3 + \beta\gamma x^0)^2}{a^2} + \frac{(\bar{x}^1)^2 + (\bar{x}^2)^2}{b^2} = 1, \tag{10}$$

where $a = \gamma x^0$ and $b = x^0$. Thus S wave front events at time t lie spatially on an ellipsoid of revolution in $\bar{S}$ with $\bar{x}^3$ the axis of symmetry, the right-hand focus at the origin of $\bar{S}$, and the center at $(0,0,-\beta\gamma x^0)$, the location of the origin of S at the dilated time $\bar{t} = \gamma t$ corresponding to the proper time t measured by a clock fixed at O. The ellipsoid is shown in Fig. 1 for $\beta = 1/\sqrt{2}$. The events occur in time from right to left in the figure and are spatially situated symmetrically about the origin of S at $\bar{t} = \gamma t$ and not the origin of $\bar{S}$.

Finally, it should be noted that the $\bar{S}$ spacetime coordinates given by Eqs. (7)–(9) also satisfy Eq. (2). But now it is clear that the latter equation, despite its suggestive form, does not describe a spherical wave front at time $\bar{t}$ because, according to Eq. (7), the wave front events in S at time t are not simultaneous in $\bar{S}$, and there is no common time $\bar{t}$ among these events to define a radius $\bar{x}^0 = c\bar{t}$. Instead, Eq. (2) expresses the fact that the ellipsoid of events in $\bar{S}$ are on the light cone. But they are not simultaneous on the light cone and so do not form a wave front in $\bar{S}$.

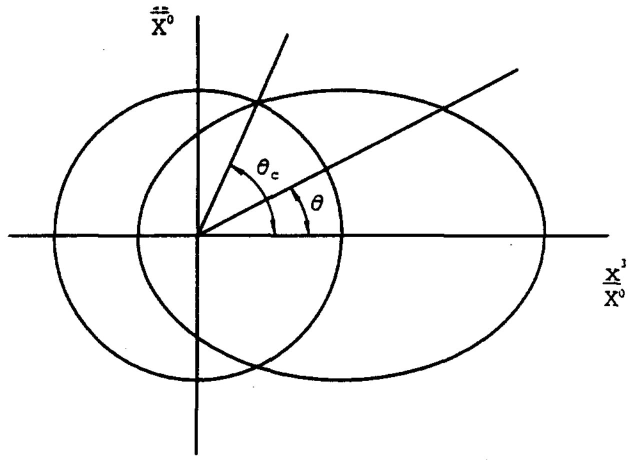

Fig. 2. An x^1, x^3 plane cross section of the object wave front and the corresponding ellipsoid of image wave front events in the S frame for $\beta = 1/\sqrt{2}$. Each image event occurs as the S wave front crosses its spatial location.

III. FORMATION OF AN ELLIPSOID OF $\bar{S}$ WAVE FRONT EVENTS IN S

There exists a corresponding ellipsoid of events in S that make up a wave front in $\bar{S}$, and I now examine in detail how it is formed. To facilitate the discussion, I will refer to the wave front in S at some arbitrary given time $t = T$ as the object wave front, the corresponding one in $\bar{S}$ as the image wave front, and the events on these respective wave fronts as object and image events. The wave front in S at any other time t will be called simply the S wave front and similarly in $\bar{S}$. While this terminology has some logical justification, as we shall see, it is used here only to provide fluency in the discussion and is certainly not meant to imply the primacy of any frame.

Now let us address the question: given a set of object wave front events $(X^0 \equiv cT, \theta, \phi)$, what are the S wave front events (x^0, θ, ϕ) that make up the corresponding image wave front in $\bar{S}$ and how are they related to the former? First of all, by the relativity principle, we expect that the corresponding image wave front should occur at the same time, as measured in $\bar{S}$, as we are specifying in S for the object wave front, namely at $\bar{t} = T$ and $\bar{x}^0 = X^0$. There are no time dilations or length contractions here. The reasoning is simply that if the two inertial observers wait the same amount of time, numerically according to their respective clocks, from the initial event at the coincident origins, then they should record the same thing. Second, from the result of Sec. II, we expect these events to be spatially distributed on an ellipsoid in S.

From Eq. (7) with $\bar{x}^0 = x^0 = X^0$, we see for any given value of β there is a minor circle of events on the object wave front defined by the critical angle

$$\theta_c \equiv \cos^{-1}\left[\frac{1}{\beta}\left(1 - \frac{1}{\gamma}\right)\right], \tag{11}$$

which occur in $\bar{S}$ at the prescribed time for the image wave front. Only the object events on this circle are common to both the object and image wave fronts. The object events with $\theta < \theta_c$ occur too early and those with $\theta > \theta_c$ occur too late. The situation is illustrated for $\beta = 1/\sqrt{2}$ in Fig. 2

where a cross section in the x^1, x^3 plane is shown.

The events that will make up the image wave front all occur simultaneously in $\bar{S}$, but on a sequence of minor circles on the S wave front as it expands radially outward. In order to qualify, an event must satisfy two conditions: (1) it must occur in $\bar{S}$ at $\bar{x}^0=X^0$ and (2) it must occur on the S wave front. For example, consider a point on the object wave front at $\theta<\theta_c$, $\phi=0$ in Fig. 2 and follow the ray radially outward at the speed of light. The events of this sequence all occur on the S wave front, satisfying condition (2), and they occur later and later as we go out in radius until one of them satisfies condition (1) as well. This event is the image of the event on the object wave front, in a reasonable usage of the word image, since in optics one follows a ray from object point to image point. For $\theta>\theta_c$ we reverse the sequence and follow the ray back in time from object event to image event.

According to Eq. (4) with $\bar{x}^0=X^0$ and the last of Eqs. (3), the two conditions require that all image events situated on rays lying on the cone given by θ, $0\leqslant\phi\leqslant 2\pi$ must occur at a time in S given by

$$x^0=\frac{X^0}{\gamma(1-\beta\cos\theta)}. \tag{12}$$

Then, substituting Eq. (12) into Eqs. (3), we have for the three rectangular spatial coordinates of the image events

$$x^1=\frac{X^0\sin\theta\cos\phi}{\gamma(1-\beta\cos\theta)}, \tag{13}$$

$$x^2=\frac{X^0\sin\theta\sin\phi}{\gamma(1-\beta\cos\theta)}, \tag{14}$$

$$x^3=\frac{X^0\cos\theta}{\gamma(1-\beta\cos\theta)}. \tag{15}$$

Equations (12)–(15) give the S frame spacetime coordinates of the image events that make up the image wave front in $\bar{S}$. Substituting Eqs. (12)–(15) into the Lorentz transformation, Eqs. (4)–(6), we obtain

$$\bar{x}^0=X^0,\quad \bar{x}^1=X^0\sin\bar{\theta}\cos\bar{\phi},$$
$$\bar{x}^2=X^0\sin\bar{\theta}\sin\bar{\phi},\quad \bar{x}^3=X^0\cos\bar{\theta}, \tag{16}$$

where

$$\sin\bar{\theta}\equiv\frac{\sin\theta}{\gamma(1-\beta\cos\theta)},\quad \cos\bar{\theta}\equiv\frac{\cos\theta-\beta}{1-\beta\cos\theta},\quad \bar{\phi}\equiv\phi, \tag{17}$$

$\bar{\theta}$ and $\bar{\phi}$ being the corresponding spherical angles in $\bar{S}$. Thus it is clear that, although the image wave front is not composed of object wave front events (except for those on the critical minor circle), it is nevertheless a sphere of radius $\bar{x}^0=X^0=cT$ in $\bar{S}$ at time $\bar{t}=T$ centered on the origin of the barred frame. As time progresses in the respective frames, object and image wave fronts expand radially at the speed of light.

It is straight forward to show that the spatial coordinates of the image events in S satisfy the equation

$$\frac{(x^3-\beta\gamma X^0)^2}{a^2}+\frac{(x^1)^2+(x^2)^2}{b^2}=1, \tag{18}$$

where again $a=\gamma X^0$ and $b=X^0$. The easiest way to do this is to first express Eqs. (13)–(15) in terms of $\bar{\theta}$ and $\bar{\phi}$ using

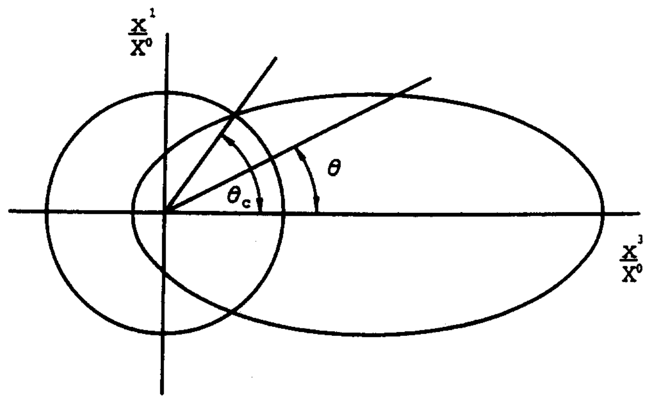

Fig. 3. An x^1, x^3 plane cross section of the object wave front and the corresponding ellipsoid of image wave front events in the S frame for $\beta=\sqrt{3}/2$. As β increases the ellipsoid elongates with the minor axis remaining constant at the radius of the object wave front.

Eqs. (17) which can be inverted by letting $\beta\rightarrow-\beta$. Equation (18) describes an ellipsoid of revolution with the following properties for all β: (1) the left-hand focus is located at the origin of S; (2) the center is located at $(0,0,\beta\gamma X^0)$ and coincides with the origin of $\bar{S}$ at the instant $\bar{t}=T$ of the image wave front as measured by a clock at $\bar{O}$; (3) the radius of the image wave front X^0 is equal to the semiminor axis and also to a length contraction of the semimajor axis. For larger values of β the ellipsoid elongates with the semiminor axis remaining constant. Figure 2 shows an x^1, x^3 plane cross section of the ellipsoid for $\beta=1/\sqrt{2}$ while Fig. 3 shows the same thing for $\beta=\sqrt{3}/2$.

The second property can be interpreted in terms of time dilation. Consider two events at the origin of $\bar{S}$, the first at $\bar{t}=t=0$ when the origins coincide, and the second at the time of the image wave front $\bar{t}=T$. Due to time dilation, the time between these two events as recorded by the clocks in S is γT, and during this time interval the $\bar{S}$ origin moves a distance $v\gamma T=\beta\gamma X^0$. Despite first impressions, in Figs. 2 and 3 the $\bar{S}$ origin at the time $t=\gamma T$ is inside the S wave front because at this time its radius is γX^0 which is always greater than $\beta\gamma X^0$.

That the ellipsoid of image events in S contracts to a sphere in $\bar{S}$ can be understood as follows. Consider the two events at the opposite ends of the major axis. In S these two events occur a distance $2\gamma X^0$ apart and at different times. In $\bar{S}$ they occur simultaneously a distance $2X^0$ apart. Consider the major axis in S to be a rigid rod. Since the rod is stationary in S, the two events occur at its opposite ends, even though they do not occur simultaneously. Thus the rod has a proper length $L_0=2a=2\gamma X^0$. In $\bar{S}$ the rod is moving, and therefore its length L is interpreted as the distance between the two events that occur simultaneously at its opposite ends. The well-known length contraction formula gives $L=L_0/\gamma=2X^0$. The same argument applies to any pair of points at opposite ends of a chord of the ellipsoid parallel to the x^3 axis. Thus each such chord is contracted by the same factor of $1/\gamma$ as the major axis. It then follows that the ellipsoid of revolution contracts into a sphere with a diameter equal to the minor axis.

IV. CONCLUSION

It is now quite clear that the ellipsoid of revolution in Fig. 1 has the same relationship with the $\bar{S}$ wave front as that of Fig. 2 with the S wave front. Indeed if one substitutes the inversions of Eqs. (17) into Eqs. (7)–(9), one obtains the form of Eqs. (12)–(15) with $\beta \rightarrow -\beta$. The ellipsoid of Fig. 1 is the spatial locus of object wave front events in $\bar{S}$, and the ellipsoid of Fig. 2 is the spatial locus of image wave front events in S. They are, respectively, made up of minor circles of $\bar{S}$ wave front events and S wave front events as these two spherical wave fronts expand radially at the speed of light from their respective origins. Thus, while the relativity principle is not violated, the relationship between the S and $\bar{S}$ wave fronts is a bit more subtle than just a Lorentz transformation. It is only by the time delays and advances on rays from the object wave front that one obtains a set of image events in S that are symmetrically distributed about the origin of $\bar{S}$. Then, upon relabeling by a Lorentz transformation, the image events become simultaneous wave front events in $\bar{S}$ and the ellipsoid is contracted into a sphere centered on the origin $\bar{O}$.

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Micro-Raman spectroscopy in the undergraduate research laboratory

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Modern materials science requires processing and characterization techniques for microscopic structures. Molecular probes such as Raman spectroscopy are some of the most viable tools, particularly if they are supplemented by imaging to obtain spatially resolved compositional information of inhomogeneous or low volume samples. In order to introduce these techniques and materials science experiments into the advanced undergraduate laboratory, we have constructed an inexpensive micro-Raman attachment, which consists of an off-the-shelf microscope and the coupling optics to an existing Raman spectrometer. The modification of the microscope, the optical coupling, and a low cost viewing system for positioning the laser excitation on the sample are described in detail. The students study molecular spectra of new materials such as diamond films, Fullerenes, and biological compounds with spatial resolution of several microns.

I. INTRODUCTION

Raman scattering—or scattering of light at altered frequency as first described by Raman and Krishnan[1]—yields structural and dynamic information on a molecular level.[2,3] As a probe it is nondestructive and therefore it is one of the most important tools for characterization of new materials. Due to recent simplification in the design of modern Raman detection systems,[4] they are becoming an option for the budget of an advanced undergraduate research laboratory. However, a search through the volumes of this journal during the last decade shows very few publications concerned with the application of modern spectroscopic techniques, and only a note which deals with Raman scattering.[5] In this paper we present Raman spectroscopic experiments to introduce students to light scattering techniques and to state of the art applications in materials science. We describe a setup, which has the advantage that the laser spot on the sample can be imaged *in situ* and that microscopic regions of a material under study can be easily probed.

In a typical Raman experiment, the excitation source is a laser, and the scattered light is analyzed by a spectrometer and a detector with sensitivity near the single photon level. The inelastically scattered light contains information on vibrational states of the sample, which manifests itself by a frequency shift from the incident light. The underlying physics is that vibrations (or other excitations) modulate the polarizability tensor and cause the induced dipole moment to radiate at frequencies different from the electric field vector of the incoming light wave. For most applications the spontaneous Raman scattering originating from the linear response to the electric field is measured. The

