## Equipartition for a relativistic gas

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In a recent paper, Menon and Agrawal[1] derived what they call the "relativistic temperature"

$$kT = \langle c^2p^2/E\rangle \quad (1)$$

of an equilibrium gas of molecules of typical instantaneous momenta $(p_1, p_2, p_3)$ and energy $E$. The brackets indicate a canonical ensemble average. A one-dimensional model was used to generalize in this way the classical result

$$kT = \langle p_j^2/2m_0\rangle, \quad (2)$$

where $m_0$ is the rest mass of a molecule. They remark with respect to Eq. (2) "that no text book answers one important question, viz. 'What is the corresponding rest frame, kinetic expression for $T$ for a gas of relativistic atoms'...". Textbook writers can amuse themselves in trying to discover whether or not their book contains Eq. (1). (Of course the result may be given only implicitly.)

In my own case, Eq. (1) does appear explicitly in a problem and as an application of a generalized equipartition theorem due to Tolman,[2] whose paper already contains Eq. (1)—remarkably enough 73 years ago.

This useful and simple result may be stated as follows. Let the Hamiltonian function $E$ depend on variables $u_1$, $u_2$,..., where a $u_i$ can be a generalized coordinate or momentum. Let the integration over $u_r$ proceed from $a$ to $b$ where either $a = 0$ or $E$ is infinite at $u_r = a$ or both; and where $b$ is subject to the same conditions. Then the theorem asserts

$$\left\langle u_r \frac{\partial E}{\partial u_r}\right\rangle = kT. \quad (3)$$

Of course the conditions on $a$ and $b$ cannot both be fulfilled for a $u_r$ that does not appear in the expression for $E$, and indeed for such a variable $u_r$ the left-hand side of Eq. (3) is zero and the theorem does not hold.

If one integrates over a momentum component $u_r = p_x$, say, from $p_x = a = 0$ to $p_x = b = \infty$, the conditions of the theorem are fulfilled. With $E = (p_x^2 + p_y^2 + p_z^2)/2m_0$, (3) yields (2). Indeed for three dimensions

$$\langle p^2/2m_0\rangle = 3kT.$$

With

$$E = (p_x^2 + p_y^2 + p_z^2 + m_0^2c^2)^{1/2}c,$$

for a gas of relativistically moving particles, and integrating over the same range, the theorem yields (1) in the form

$$\langle c^2p_x^2/E\rangle = kT$$

It holds for any number of dimensions, and I implicitly drew attention to this apparently little known result some years ago in a different context.[4]

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## Nonlocality in frequency measurements of uniformly accelerating observers

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The standard extension of special relativity to accelerated frames of reference is based upon the assumption that an accelerated observer is equivalent to an instantaneously comoving inertial observer. While this hypothesis of locality[1] is exactly valid for pointlike coincidences as in Newtonian mechanics, it is only approximately valid for observations by an accelerating observer that are not limited to a single point in space-time. The measurement of the frequency of a wave associated with a particle by an accelerating observer is an example of such a nonlocal observation.[2] Neglect of the limitations of the hypothesis of locality for photons has led to results in the literature for the relativistic Doppler shift for accelerating observers being regarded as exact, when, in fact, they are not. They should be qualified by the reservation, "if the distance and time scales over which nonlocal measurements are made are negligible."

For example, the following equation for the relativistic Doppler shift of the frequency of a photon measured by an observer in an accelerating frame has been derived by Price[3] and confirmed by Landsberg and Bishop[4] as exact:

$$\nu'/\nu = 1 - gh/c^2, \tag{1}$$

where $\nu'$ and $\nu$ are the frequencies of the photon measured in an accelerating frame S′ and an inertial frame S, respectively, $g$ is the constant acceleration of the origin of S′ with respect to S, and $h$ is the rod distance between the source and the detector. Due to the limitations of the hypothesis of locality, Eq. (1) is strictly valid only for distance and time scales satisfying the conditions $gh/c^2 \ll 1$ and $g/c\nu \ll 1$.

Assuming the equivalence of an accelerating observer and an instantaneously comoving inertial observer, the above authors used the standard relativistic Doppler shift equation[5]

$$\frac{\nu'}{\nu} = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} \tag{2}$$

in deriving Eq. (1). But between the times of arrival at the observation point in S′ of two successive crests of the wave associated with the photon, the accelerating frame is not moving at constant velocity $v$ with respect to the inertial frame as assumed in Eq. (2).

In a couple of more recent papers Hamilton[6] and Cochran[7] are each in agreement with an expression given by

$$\nu'/\nu = \exp(-g\tau_1/c), \tag{3}$$

where $\tau_1$ is the proper time when an accelerated observer in S′ receives the first of two successive light flashes (or wavecrests) generated by a stationary source in S. In his derivation of Eq. (3) Cochran used the standard time dilation from special relativity,

$$\Delta\tau = \Delta t(1 - v^2/c^2)^{1/2}, \tag{4}$$

where $\Delta\tau = \tau_2 - \tau_1$ and $\Delta t = t_2 - t_1$. Again, contrary to the assumption in Eq. (4), the accelerated observer is not moving at constant velocity $v$ with respect to S during the time interval between the two events.

In this note we consider a light wave instead of the world line of a photon to derive an exact result for the Doppler shift which reduces to Eqs. (1) and (3) for $gh/c^2 \ll 1$ and $g/c\nu \ll 1$. Quantum aspects of this problem for single photons are discussed elsewhere.[2] Our approach is to relate the space-time coordinates in S and S′ of the two events mentioned above. We use the coordinate transformation equations for hyperbolic motion.[8] We thus go back to basics and derive a relativistic Doppler shift equation specifically for accelerated motion instead of using equations that were previously derived for another situation.

Consider an inertial Lorentz frame S and a noninertial frame S′ the origin O′ of which is accelerating with respect to S at a constant rate $g$ in a direction along their common $x$ axes as shown in Fig. 1. With an appropriate choice of the origin O for S, the world line of O′ with respect to S is one branch of the hyperbola[8]

$$x^2 - c^2t^2 = c^4/g^2. \tag{5}$$

Define a set of primed coordinates for S′ based upon a Fermi–Walker transported tetrad at O′. The coordinate transformation is given by[9]

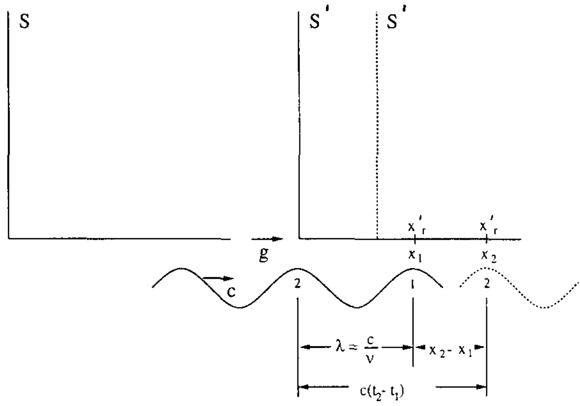


Fig. 1. The S′ frame is accelerating at a constant rate $g$ with respect to an inertial frame S in the direction of their common $x$ axis. A light wave is traveling in the $x$ direction. A receiver is fixed in S′ at $x' = x'_r$. The space-time coordinates of two events are recorded that are the passage of wavecrests 1 and 2 by the receiver. The S′ frame and the wave at the second event are shown by dashed lines. The unprimed and primed coordinates for the two events are denoted by $(t_1,x_1)$, $(t_2,x_2)$, $(t'_1,x'_r)$, and $(t'_2,x'_r)$. The wavelength $\lambda = c/\nu$ in S, plus the distance between the two events, $x_2 - x_1$, is equal to the distance traveled by the second wavecrest in the time between the two events $c(t_2 - t_1)$.

$$t = \left(\frac{x'}{c} + \frac{c}{g}\right)\sinh\frac{gt'}{c}, \tag{6}$$

$$x = \left(x' + \frac{c^2}{g}\right)\cosh\frac{gt'}{c}, \tag{7}$$

along with $y = y'$ and $z = z'$. The line element expressed in the prime coordinates is[9]

$$ds^2 = -(1 + gx'/c^2)c^2\,dt'^2 + dx'^2 + dy'^2 + dz'^2, \tag{8}$$

and coordinate time $t'$ is kept by a clock at the origin O′. Thus coordinate time $t'$ is equivalent to proper time $\tau$ along the hyperbolic world line of O′ in S.

We now use the coordinate transformation equations (6) and (7) to derive the Doppler shift between the frequency of a wave observed in S and that observed in S′. Suppose a receiver is fixed in the S′ frame at $x' = x'_r$. Referring to Fig. 1, define $t_1$ and $t_2$ as the times, as measured in S, when two successive wavecrests are at the receiver. The unprimed space coordinates of these two events are defined to be $x_1$ and $x_2$, while the common primed space coordinate of both events is by definition $x'_r$. The corresponding primed time coordinates are $t'_1$ and $t'_2$.

As seen from Fig. 1, the wavelength $\lambda = c/\nu$, as measured in S, plus the distance moved by the receiver in the time between crests, $x_2 - x_1$, is equal to the distance traveled by the second crest in this same time:

$$c/\nu + (x_2 - x_1) = c(t_2 - t_1). \tag{9}$$

Transforming Eq. (9) to prime coordinates and converting the hyperbolic functions to exponentials, we obtain

$$t'_2 - t'_1 = -\frac{c}{g}\ln\left(1 - \frac{g e^{(gt'_1/c)}}{c\nu(1 + gx'_r/c^2)}\right). \tag{10}$$

It is tempting to say that $t'_2 - t'_1$ in Eq. (10) is the period of the light wave measured at $x'_r$ in S′, but Eq. (10) gives the time interval between wavecrests in terms of coordinate time kept by a clock at the origin. The period $T'$ that an observer at $x'_r$ measures is the proper time interval measured by a fixed clock at $x'_r$. From Eq. (8) with $ds^2 = -c^2 d\tau^2$ for a timelike interval and $dx' = dy' = dz' = 0$, the proper time interval between the arrival of the two crests at $x' = x'_r$ is given by

$$T' = \tau_2 - \tau_1 = (1 + gx'_r/c^2)(t'_2 - t'_1). \quad (11)$$

Eliminating $t'_2 - t'_1$ from Eqs. (10) and (11), we have

$$T' = \frac{1}{\nu'} = -\frac{c}{g}\left(1 + \frac{gx'_r}{c^2}\right)\ln\left(1 - \frac{g e^{(gt'_1/c)}}{c\nu(1 + gx'_r/c^2)}\right). \quad (12)$$

Equation (12) is an exact expression for the period $T'$ and the frequency $\nu'$ of the light wave as measured by an observer at a fixed point $x'_r$ in the accelerated frame S' at a given coordinate time $t'_1$ for the passage of the first crest. For $gx'_r/c^2 \ll 1$ and $g/c\nu \ll 1$ and noting that under these conditions the coordinate time $t'_1$ is equivalent to the proper time $\tau_1$, Eq. (12) reduces to Eq. (3) in agreement with Hamilton[6] and Cochran.[7]

In order to compare with Eq. (1), which does not depend upon time explicitly, we note that for a given light wave the observation time $t'_1$ and the observation position $x'_r$ are not independent. From Eq. (8) with $ds^2 = 0$, the world line for a wavecrest moving in the $x'$ direction is given by

$$(c/g)\ln(1 + gx'/c^2) = t' + \text{const.} \quad (13)$$

Set the zero of coordinate time such that the first wavecrest is at the origin $x' = 0$ at coordinate time $t = t' = 0$. Then Eq. (13) with the integration constant set to zero gives the world line of the first wavecrest. By definition at $t' = t'_1$ the first wavecrest is at $x' = x'_r$, so we have

$$e^{(gt'_1/c)} = 1 + gx'_r/c^2. \quad (14)$$

Substituting Eq. (14) into Eq. (12), we obtain an exact expression for the Doppler shifted frequency of the given light wave as measured by an observer at $x'_r$ in the accelerated frame S':

$$\nu' = \frac{1}{T'} = -\frac{g}{c}\left(1 + \frac{gx'_r}{c^2}\right)^{-1}\left[\ln\left(1 - \frac{g}{c\nu}\right)\right]^{-1}. \quad (15)$$

The logarithms in Eqs. (12) and (15) are due to the acceleration of the receiver between the arrival times of the two successive wavecrests. For a given rate of acceleration $g$ this effect becomes more pronounced for lower frequencies because then there is more time for the receiver to change its state of motion between crests.

For $g/c\nu \ll 1$ and $gx'_r/c^2 \ll 1$, Eq. (15) reduces to Eq. (1) with $x'_r = h$, that is with the source of Refs. 3 and 4 located at the origin O'. The distance $h$ between the photon source and the detector corresponds in the present treatment of a wave to the first wavecrest being at the origin O' at coordinate time zero and the receiver being fixed at $x' = x'_r$. In the present context, however, the position of a wave source with respect to the receiver is immaterial. There simply exists a wave and its frequency can be measured by observers in S and S'. All observers in S will measure the same frequency $\nu$, but observers in S' will measure frequencies $\nu'(x')$ depending on the observer's position $x'$.

We now turn our attention briefly to the gravitational redshift[10] which is readily obtained as the difference of a sequence of Doppler shifts observed on the same two successive wavecrests at two different points in S'. Consider the two successive wavecrests to be part of a short wavetrain. As they leave O' and travel out the $x'$ axis they are detected first by an observer at $x'_1$ and later by another observer at $x'_2$ with $x'_2 > x'_1$. According to Eq. (15) the observers at $x'_1$ and $x'_2$ will record frequencies given, respectively, by

$$\nu'(x'_i) = -\frac{g}{c}\left(1 + \frac{gx'_i}{c^2}\right)^{-1}\left[\ln\left(1 - \frac{g}{c\nu}\right)\right]^{-1} \quad (i = 1,2). \quad (16)$$

Then the fractional frequency shift between the frequency measurements by observers at $x'_1$ and at $x'_2$ is

$$\frac{\Delta\nu'}{\nu'} = \frac{\nu'(x'_2) - \nu'(x'_1)}{\nu'(x'_1)} = -\frac{(g/c^2)(x'_2 - x'_1)}{1 + gx'_2/c^2}. \quad (17)$$

Equation (17) is an exact result for the gravitational redshift in the accelerated frame. The nonlocality factor has canceled in taking the ratio. Because the motion is at constant acceleration the nonlocal aspects of the frequency measurements at $x'_1$ and $x'_2$ are the same. Consequently, Eq. (17) written in the form $\nu'(x'_2)/\nu'(x'_1)$ agrees exactly with a corresponding result derived by Cochran[11] using the line element Eq. (8) and assuming locality.

The canceling of the nonlocality factor and the agreement of Eq. (17) with Cochran's result indicate that nonlocality is not significant in the gravitational redshift for uniformly accelerated observers. This result raises the question, what about nonuniformly accelerated observers, or more importantly, what about stationary observers in a real gravity field such as Schwarzschild that are accelerating nonuniformly with respect to local inertial frames. While nonuniform acceleration is beyond the scope of the present work, we note that if the nonlocality factors in Eq. (16) depended on position in S' they would not cancel in taking the frequency ratio.[12]

Although waves in noninertial frames have been considered previously,[2] to the author's knowledge the exact results given by Eqs. (12) and (15) for the relativistic Doppler shift for uniformly accelerating observers are new.

Deviations from local behavior depend on the distance and time scales, $gx/c^2$ and $g/c\nu$, over which nonlocal measurements are made. For $x$ of the order of laboratory dimensions and $\nu$ of radio frequency or greater, these scales are insignificant even for accelerations that can be produced by the latest ultracentrifuge technology. On the other hand an $Fe^{57}$ nucleus in a crystal lattice is subjected to approximately periodic accelerations of amplitude of the order of $10^{17}$ m s$^{-2}$ due to lattice vibrations, giving $g/c\nu \cong 10^{-10}$ for the 14.4-keV Mössbauer transition. While the frequency of this transition has an uncertainty $\Delta\nu/\nu$ of the order of $10^{-13}$, no acceleration-dependent shift in the frequency has been observed.[13] The lifetime of the excited state is long compared to the period of the vibration, and the transition takes place over many cycles. Consequently, effects that are linear in the velocity and the acceleration are cancelled, and only the second-order

Doppler shift which is proportional to temperature is observed.[14] What is needed is an acceleration that is comparable in magnitude to that provided periodically by lattice vibrations but sustained over a longer period of time. Such accelerations can be realized by bombarding a target with heavy ions in a heavy ion accelerator. But then the projectile is not really part of the lattice and the recoiless transition is lost.

For the moment effects of the nonlocal nature of frequency measurement seem to be beyond the access of experiment. Nevertheless, from a conceptual point of view it is important to understand the limitations of standard results such as Eqs. (1) and (3).

## ACKNOWLEDGMENT

I would like to thank the referees for some stimulating criticism.

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[9] See Ref. 8, pp. 172–173.

[10] The term gravitational redshift is used here even though the accelerating frame exists in flat space-time. In the spirit of the equivalence principle such usage is fairly widespread. See for example Refs. 3, 4, 6, and 7.

[11] See Ref. 7, Eq. (22).

[12] The case of nonuniform acceleration is of course not as simple as letting $g$ go to $g(x')$ in the results for uniform acceleration. The linear differential equations of hyperbolic motion become nonlinear in the case of nonuniform acceleration.

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### EINSTEIN'S PEGASUS

There's Einstein riding on a ray of light,
Holding a mirror up, at arm's length in his hand,
In which he cannot see his face in flight
Because his jesting image, I now understand.

Won't ever reach the mirror since its speed,
Too, is the speed of light. He rides, this fleeting day,
As if on Pegasus, immortal steed
Of bridled meditation, past the Milky Way,

Out to my mind's Andromeda, where I,
Also transported, staring at a windless pool,
Watch his repaired reflection whizzing by.
Thought he can't see himself, this self-effacing fool

Who holds all motion steady in his head,
I won't forget his facing what he cannot see
In thought that binds the living and the dead,
And ride with him, outfacing fixed eternity.

Robert Pack, Middlebury College, 1991.