on account of Gauss' theorem. If the field **F** vanishes sufficiently rapidly at large distances, so that the integral in Eq. (A3) exists, then both terms in Eq. (A10) vanish and the proof ends.

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[12]See Ref. 6, p. 47.

[13]For a brief discussion on the Maxwell equations with magnetic monopoles see, for example, D. J. Griffiths, *Introduction to Electrodynamics* 2nd ed. (Prentice–Hall, Englewood Cliffs, NJ, 1989) Sec. 7.3.3; see also J. D. Jackson in Ref. 9; Sec. 6.12. For a nice but somewhat arguable deduction of Maxwell's equations with magnetic monopoles, see the recent paper of F. S. Crawford, "Magnetic monopoles, Galilean invariance, and Maxwell's equations," Am. J. Phys. **60**, 109–114 (1992); and the subsequent two related papers: A. Horzela, E. Kapuscik, and C. A. Uzes, "Comment on 'Magnetic monopoles, Galilean invariance, and Maxwell's equations,' by F. S. Crawford," Am. J. Phys. **61**, 471–472 (1993); F. S. Crawford, "A response to A. Horzela, E. Kapuscik, and C. A. Uzes 'Comment on Magnetic monopoles, Galilean invariance, and Maxwell's equations.' " *ibid.* **61**, 472 (1993).

[14]Obviously not all the time-dependent vector field propagate hyperbolically. If the sign (+) preceding the last term in Eq. (13) is changed by (−) then the field **F** propagates elliptically in vacuum with speed *c*.

[15]It can be argued, however, that if the definition of retarded potentials is used, then the formulas (18) and (19) become $\mathbf{E}=-\nabla\Phi_e-\nabla\times\mathbf{A}_m-\partial\mathbf{A}_e/\partial t$ and $\mathbf{B}=-\nabla\Phi_m+\nabla\times\mathbf{A}_e-(1/c^2)\partial\mathbf{A}_m/\partial t$, (with *e* and *m* denoting electric and magnetic). These expressions are known in the literature of magnetic monopoles. However, the approach followed here for obtaining Eqs. (18) and (19) is relatively simpler than the known approach based on potentials—recalling that in this case there are two pairs of potentials $\{\Phi_e, \mathbf{A}_e\}$ and $\{\Phi_m, \mathbf{A}_m\}$ so that the derivation of **E** and **B** in terms of potentials does not appear to be as simple as one would like [for such a derivation, one can use the Helmholtz theorem for antisymmetric tensors in (3+1) space time; see, e.g., D. H. Kobe, "Helmholtz's theorem for antisymmetric second-rank tensor fields and electromagnetism with magnetic monopoles," Am. J. Phys. **54**, 354–358 (1984); J. A. Heras, "A short proof of the generalized Helmholtz theorem," *ibid.* **58**, 154–155 (1990)].

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[17]See, for example, L. Eyges in Ref. 9 or also A. R. Janah *et al.* in Ref. 8.

Relativistic (an)harmonic oscillator

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The relativistic extension of one-dimensional simple harmonic motion is developed in the Lagrangian formalism. The relativistic equations of motion are derived and solved analytically. The motion with respect to proper time is analyzed in terms of an effective potential energy. While the motion remains bounded and periodic, the effect of time dilation along the world line is to cause it to become anharmonic with the period increasing with amplitude and the curvature concentrated at the turning points.

I. INTRODUCTION

A particle undergoing constant acceleration and the simple harmonic oscillator are two elementary topics in classical mechanics that are thoroughly discussed in all of the standard expositions of the subject.[1] But while the relativistic generalization of constant acceleration, defined with respect to instantaneously comoving inertial frames, has received a complete treatment in the literature[2–4] the relativistic extension of simple harmonic motion,[5–7] by comparison, is somewhat incomplete. Synge[5] has given an exact expression for the period of terms of an integral which Goldstein[6] has identified as being expressible in terms of standard elliptical integrals. Skinner[7] covers the topic in a problem. All three of these authors give the relativistic correction to the period to leading order. But they have not calculated the world line nor analyzed the motion in any detail.

The question, "What happens to a simple harmonic oscillator when the energy is such that the velocities become relativistic?," is a natural one to ask, and in this paper we give a complete answer. In brief, the effect of time dilation along the world line is to cause simple harmonic motion at low energy to become anharmonic at high energy, and hence the parentheses around the "an" in our title. Since the simple harmonic oscillator is a central idea in physics, from normal modes of vibration in molecules and solids to those in relativistic quantum fields, it is worthwhile considering a complete and rigorous relativistic treatment of the basic classical model.

II. THE RELATIVISTIC EQUATIONS OF MOTION

We consider the relativistic motion of a particle of rest mass m_0 in a one-dimensional harmonic oscillator potential, $\frac{1}{2}kx^2$. We restrict our attention to one-dimensional motion in a (1+1) dimensional Minkowski space with coordinates $x^0=ct$ and $x^1=x$, a metric tensor $\eta=\text{diag}(-1,+1)$, and a line element for timelike intervals along the particle's world line given by

$$-c^2d\tau^2=-c^2\left(\frac{d\tau}{d\lambda}\right)^2 d\lambda^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda^2=-\left(\frac{dx^0}{d\lambda}\right)^2 d\lambda^2+\left(\frac{dx^1}{d\lambda}\right)^2 d\lambda^2, \tag{1}$$

where τ is a proper time and λ is any general evolution parameter.

Following Synge,[5] we construct a relativistic Lagrangian for our system by appending an appropriate potential term to the geometric Lagrangian for a free particle[8]

$$L=-m_0c\sqrt{-\dot{x}_\nu\dot{x}^\nu}-\frac{1}{2}k(x^1)^2\frac{\dot{x}^0}{c}, \tag{2}$$

where the dots denote derivatives with respect to λ. This expression is not Lorentz invariant and we are using a special frame of reference. The equations of motion are obtained from the Euler–Lagrange equations for stationary action

$$\frac{d}{d\lambda}\frac{\partial L}{\partial\dot{x}_\mu}-\frac{\partial L}{\partial x_\mu}=0, \quad \mu=0,1. \tag{3}$$

We first consider the equations of motion generated by Eqs. (2) and (3) for $\lambda=t$, the coordinate time. In this case

$$\dot{x}^0=dx^0/dt=c \tag{4}$$

and

$$-\dot{x}_\nu\dot{x}^\nu=-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=(\dot{x}^0)^2-(\dot{x}^1)^2=c^2-(\dot{x}^1)^2. \tag{5}$$

As $\partial L/\partial x_0=0$, the coordinate x_0 is cyclic, and the corresponding Euler–Lagrange equation, Eq. (3) with $\mu=0$ and $\lambda=t$, indicates that

$$\frac{\partial L}{\partial\dot{x}_0}=\frac{1}{c}\left(\frac{m_0c^2}{\sqrt{1-(\dot{x}^1)^2/c^2}}+\frac{1}{2}k(x^1)^2\right)=\frac{E}{c} \tag{6}$$

is a constant of the motion, where E is the total relativistic energy, including the potential energy. Note that $\dot{x}_0=-\dot{x}^0$ and Eqs. (4) and (5) have been substituted after partially differentiating L with respect to $\dot{x}_0$. Thus we see that the factor $\dot{x}^0/c$ in the potential energy term of the Lagrangian is required in order that potential energy be included in the conservation of total relativistic energy.

Similarly, the other Euler–Lagrange equation, Eq. (3) with $\mu=1$, yields the equation of motion,

$$\frac{d}{dt}\left(\frac{m_0\dot{x}^1}{\sqrt{1-(\dot{x}^1)^2/c^2}}\right)+kx^1=0, \tag{7}$$

which agrees with previously published results.[5,7] Equation (7) is also obtainable from Newton's equation of motion in the form, $dp/dt+kx^1=0$, by replacing the Newtonian momentum with its coordinate-time relativistic form.

An analytic solution of Eq. (7) is possible, although not readily apparent. We prefer to work with the proper-time equations of motion for which $\lambda=\tau$. A complete analytic solution is easily obtained, and in this formalism the relativistic motion can be analyzed by means of an effective potential energy which we will define subsequently.

The proper-time equations of motion are obtained from Eqs. (2) and (3) with $\lambda=\tau$, and now the dots denote derivatives with respect to proper time. In this case, from Eq. (1) with $\lambda=\tau$, we have

$$\dot{x}^0=\frac{dx^0}{d\tau}=\frac{dt}{d\tau}\frac{dx^0}{dt}=\frac{c}{\sqrt{1-(dx^1/dt)^2/c^2}}, \tag{8}$$

and

$$-\dot{x}_\nu\dot{x}^\nu=-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=c^2. \tag{9}$$

The coordinate x_0 is still cyclic, and Eq. (3) with $\mu=0$ and $\lambda=\tau$ indicates that

$$\frac{\partial L}{\partial\dot{x}_0}=m_0\dot{x}^0+\frac{1}{2}\frac{k}{c}(x^1)^2=\frac{E}{c} \tag{10}$$

is constant, where we have used Eq. (9) to obtain the first equality, Eq. (8) for the second, and E is the total relativistic energy given by Eq. (6).

Substituting $\partial L/\partial\dot{x}_0$ from Eq. (10) and $\partial L/\partial x_0=0$ into Eq. (3) with $\mu=0$ and $\lambda=\tau$, we obtain

$$m_0\ddot{x}^0+\frac{k}{c}x^1\dot{x}^1=0. \tag{11}$$

For $\mu=1$ we have similarly $\partial L/\partial\dot{x}_1=m_0\dot{x}^1$ and $\partial L/\partial x_1=(k/c)x_1\dot{x}_0=-(k/c)x^1\dot{x}^0$, and substitution into Eq. (3), with $\lambda=\tau$, gives us

$$m_0\ddot{x}^1+\frac{k}{c}x^1\dot{x}^0=0. \tag{12}$$

Equations (11) and (12) are the equations of motion in terms of proper time, the former being an expression of the conservation of total relativistic energy. They are compatible with the kinematic constraints $\dot{x}\cdot\dot{x}=-c^2$ and $\dot{x}\cdot\ddot{x}=0$, and can be obtained directly from the coordinate-time equations of motion. Defining the relativistic momentum $p^\mu\equiv m_0\dot{x}^\mu$, they can be expressed compactly as

$$\dot{p}^\mu=\frac{k}{c}x^1(\epsilon^{\mu\nu}\dot{x}_\nu)\equiv F^\mu, \quad \mu=0,1, \tag{13}$$

where F^μ is a Hooke's-law Minkowski force, $\epsilon_{\mu\nu}$ are the elements of a rank two Levi–Civita antisymmetric tensor,[9] and $\epsilon^{\mu\nu}=-\epsilon_{\mu\nu}$. Using the identity, $\epsilon_{\mu\rho}\epsilon^{\mu\nu}=-\delta^\nu_\rho$, we obtain

$$F_\mu F^\mu=-\left(\frac{kx^1}{c}\right)^2\dot{x}^\rho\dot{x}_\rho=(kx^1)^2, \tag{14}$$

and taking the square root of this equation, we see that the classical Hooke's law force survives in the proper-time formalism as the magnitude of the Minkowski force.

III. SOLUTION OF THE PROPER-TIME EQUATIONS OF MOTION

Expressed in terms of $x=x^1$ and $t=x^0/c$ and setting $k=m_0\omega^2$, the proper-time equations of motion, Eqs. (12) and (10), may be written as

$$\frac{d^2x}{d\tau^2}+\omega^2 x\frac{dt}{d\tau}=0 \tag{15}$$

and

$$\frac{dt}{d\tau}+\frac{\omega^2x^2}{2c^2}=\gamma_0, \tag{16}$$

where $\gamma_0\equiv E/m_0c^2$ is the total relativistic energy in units of the rest energy. The constant γ_0 is also the value of the velocity parameter $\gamma\equiv 1/\sqrt{1-(dx/dt)^2/c^2}$ at the origin, as can be seen by evaluating Eq. (6) at $x=0$. Then Eqs. (8) and (16) give the velocity parameter as a function of x as

$$\gamma(x)\equiv\frac{dt}{d\tau}(x)=\gamma_0-\frac{\omega^2x^2}{2c^2}. \tag{17}$$

Furthermore, since $\omega^2=k/m_0$, the function $\gamma(x)-1$ is the kinetic energy $T(x)$ at displacement x in units of the rest energy:

$$\gamma(x)-1=\frac{m_0c^2+T(x)+\frac{1}{2}kx^2}{m_0c^2}-\frac{\frac{1}{2}kx^2}{m_0c^2}-1=\frac{T(x)}{m_0c^2}. \tag{18}$$

Finally, substituting Eq. (17) into Eq. (15) for $dt/d\tau$, we obtain

$$\frac{d^2x}{d\tau^2}+\omega^2x\left(\gamma_0-\frac{\omega^2x^2}{2c^2}\right)=0. \tag{19}$$

The solution of Eq. (19) determines the motion in terms of proper time for a given total relativistic energy γ_0. Because of the transcendental nature of the equation, we can only obtain the solution in the form $\tau(x)$. Then Eq. (16) determines the world line in the form $t[x,\tau(x)]$. But before solving these equations, it is conceptually useful to discuss qualitatively the effect of $\gamma(x)$ in Eq. (19) and to obtain the nonrelativistic limit of this equation. The $\gamma(x)$ factor represents the effect of time dilation, $dt/d\tau(x)$, along the world line. It is instructive to consider Eq. (19) in the form,

$$\frac{d^2x}{d\tau^2}+\omega^2\left[1+\frac{T(x)}{m_0c^2}\right]x=0, \tag{20}$$

where we have substituted Eq. (18) for $\gamma(x)$. The relativistic oscillator has an angular frequency, $\omega(x)=\omega\sqrt{1+T(x)/m_0c^2}$, that varies with the kinetic energy $T(x)$, from a maximum value of $\omega\sqrt{E/m_0c^2}$ at the origin to a minimum value of ω at the turning points $x=\pm a$. Thus the effect of time dilation is to cause the relativistic oscillator to become anharmonic, with a range of angular frequencies along its world line. The strength of the anharmonicity increases with increasing total relativistic energy, and in the nonrelativistic limit the oscillator becomes harmonic. In this limit $T(x)/m_0c^2=(1/2)v^2/c^2\ll 1$, and Eq. (20) becomes

$$\frac{d^2x}{d\tau^2}+\omega^2x+O(v^2/c^2)=0, \tag{21}$$

and the proper time becomes equivalent to the coordinate time to this order.

In seeking a solution of Eqs. (16) and (19) it is convenient to re-express them in nondimensional form:

$$\frac{d\theta}{d\Theta}=\gamma_0-\frac{1}{2}X^2, \tag{22}$$

$$\frac{d^2X}{d\Phi^2}+\gamma_0X-\frac{1}{2}X^3=0, \tag{23}$$

where $X\equiv\omega x/c$ is a nondimensional displacement and $\theta\equiv\omega t$ and $\Theta\equiv\omega\tau$ are nondimensional coordinate and proper time parameters, respectively.

Equation (23) is derivable from a Lagrangian given by

$$L=\tfrac{1}{2}\dot{X}^2-(\tfrac{1}{2}\gamma_0X^2-\tfrac{1}{8}X^4), \tag{24}$$

where $\dot{X}\equiv dX/d\Theta$. In contrast to the other Lagrangian we have used which has direct physical significance, the Lagrangian of Eq. (24) is designed only to exploit an analogy with classical mechanics in the solution of Eq. (23). Since the Lagrangian does not depend explicitly on the proper time parameter Θ and the "kinetic energy" $T=\frac{1}{2}\dot{X}^2$ is a homogeneous quadratic function of the velocity $\dot{X}$, we know that the Hamiltonian is the "total energy" and that it is a constant of the motion,[10]

$$H=T+V\equiv W=\text{const.}, \tag{25}$$

where the "potential energy" is

$$V=\tfrac{1}{2}\gamma_0X^2-\tfrac{1}{8}X^4. \tag{26}$$

Of course the "energy" with which we are concerned here is neither the usual quantity of classical mechanics nor the physical relativistic energy. In fact one can show that $W=(E^2/m_0^2c^4-1)/2$. For this reason we previously have referred to the expression given by Eq. (26) as an effective potential energy. Henceforth we will omit the quotation marks and refer to W, T, and V as effective energies. At least W is a monotonic function of E, and both are constants of the motion, although trivially related. The mathematical analogy with classical mechanics that we are exploiting here is similar to the procedure that is used to determine the radial motion in the Schwarzschild geometry in general relativity.[11] One other aspect of Eq. (26) should be noted. Unlike the usual idea of a potential energy, the effective potential energy depends upon the total relativistic energy through the factor $\gamma_0=E/m_0c^2$ in the quadratic term. Thus we have a different effective potential energy curve for different motions.

In Fig. 1 we show plots of V vs X for four values of $\beta_0=v_0/c=\sqrt{\gamma_0^2-1}/\gamma_0$. The unbounded exterior regions $|X|\gg 1$ are dominated by the negative anharmonic quartic term in the effective potential energy. However, we will see that the particle never has enough effective energy to surmount the potential barrier and reach the unbounded regions. We can write the effective kinetic energy as

$$T=\frac{1}{2}\left(\frac{d\theta}{d\Theta}\frac{dX}{d\theta}\right)^2, \tag{27}$$

and since $d\theta/d\Theta=dt/d\tau=\gamma$ and $dX/d\theta=v/c=\sqrt{\gamma^2-1}/\gamma$, we have

$$T=\tfrac{1}{2}(\gamma^2-1). \tag{28}$$

Furthermore, differentiating Eq. (26) once with respect to X and setting the result to zero, we find that the positions of the

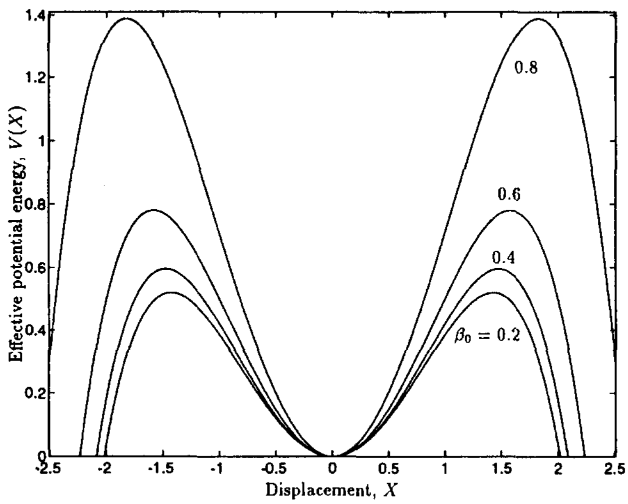

Fig. 1. The relativistic effective potential energy $V(X)$ as a function of displacement X for $\beta_0=v_0/c=0.2$–0.8. As the total energy increases, the effective potential energy maximum increases and moves further out. A massive particle never has enough effective energy to surmount the effective potential barrier and the motion is always bounded.

maximum value of the effective potential energy are given by

$$X_{V_{max}}=\pm\sqrt{2\gamma_0}, \tag{29}$$

and substituting these values back into Eq. (26), we see that

$$V_{max}=\tfrac{1}{2}\gamma_0^2. \tag{30}$$

Then from Eqs. (28) and (30) we see that the total effective energy $W=T_0\equiv T(X=0)$ is less than the maximum effective potential energy:

$$W=T_0=\tfrac{1}{2}(\gamma_0^2-1)=V_{max}-\tfrac{1}{2}<V_{max}, \tag{31}$$

and the motion is bounded, except in the limit $\gamma_0\rightarrow\infty$ which is denied to a massive particle.

At the turning points $X\equiv\pm A$ the effective kinetic energy is zero, so we can write $T_0=V(X=\pm A)$, or from Eqs. (26) and (28),

$$\tfrac{1}{2}(\gamma_0^2-1)=\tfrac{1}{2}\gamma_0 A^2-\tfrac{1}{8}A^4. \tag{32}$$

Then, solving the quartic equation, we obtain

$$A_{\pm}=\sqrt{2(\gamma_0\pm 1)}. \tag{33}$$

Equations (33) and (29) indicate that $A_+ > X_{V_{max}}$, so the physical amplitude is

$$A=A_-=\sqrt{2(\gamma_0-1)}, \tag{34}$$

and the physical motion is confined to the region $-A_-\leqslant X\leqslant A_-$.

Having established that the relativistic motion is bounded, we now proceed with the general solution. To obtain the proper time parameter Θ as a function of the displacement X of the particle, we solve the general effective energy equation,

$$\tfrac{1}{2}(\gamma_0^2-1)=\tfrac{1}{2}\dot{X}^2+\tfrac{1}{2}\gamma_0 X^2-\tfrac{1}{8}X^4, \tag{35}$$

for $\dot{X}=dX/d\Theta$, rearrange the resulting equation, and express it in the form

$$d\Theta=\pm\frac{2dX}{\sqrt{(A_+^2-X^2)(A_-^2-X^2)}}, \tag{36}$$

where the + sign is to be chosen for $dX>0$ and the $-$sign for $dX<0$, so that $d\Theta$ is always positive. By a change of variable, $X=A_-\sin\alpha$, Eq. (36) can be expressed in the standard form of an elliptic integral of the first kind.[12] Then $\Theta(X)$ is given by

$$\Theta=\omega\tau=\frac{2}{A_+}\int_0^{\phi}\frac{d\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}}\equiv\frac{2}{A_+}F(\phi,\kappa), \tag{37}$$

where $\kappa\equiv A_-/A_+=\sqrt{(\gamma_0-1)/(\gamma_0+1)}$ and $\phi\equiv\sin^{-1}(X/A_-)$.

The world line $t(x)$ or equivalently $\theta(X)$ can now be obtained from Eq. (22):

$$d\theta=(\gamma_0-\tfrac{1}{2}X^2)d\Theta. \tag{38}$$

If we substitute Eq. (36) for $d\Theta$ in Eq. (38), again make the change of variable $X=A_-\sin\alpha$, and integrate, we obtain

$$\theta=\gamma_0\Theta-(\kappa A_-)\int_0^{\phi}\frac{\sin^2\alpha\, d\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}}. \tag{39}$$

The integral in Eq. (39) can be expressed as $[F(\phi,\kappa)-E(\phi,\kappa)]/\kappa^2$, where

$$E(\phi,\kappa)\equiv\int_0^{\phi}\sqrt{1-\kappa^2\sin^2\alpha}\, d\alpha \tag{40}$$

is an elliptic integral of the second kind.[12] Making this substitution and substituting Eq. (37) for Θ, we find that

$$\theta(X)=\sqrt{2(\gamma_0+1)}E(\phi,\kappa)-\sqrt{\frac{2}{\gamma_0+1}}F(\phi,\kappa), \tag{41}$$

where $X=\sqrt{2(\gamma_0-1)}\sin\phi$ and $\kappa=\sqrt{(\gamma_0-1)/(\gamma_0+1)}$, giving the world line. The motion with respect to coordinate time is not as easy to interpret qualitatively, but we now show by calculation that the anharmonicity is present here as well.

In Fig. 2 we show plots of the normalized displacement, $X(\theta)/A_-$ vs θ for $\beta_0=v_0/c=0.20$, 0.90, and 0.99 for one complete cycle. The amplitudes for these three world lines are, respectively, 0.20, 1.61, and 3.49 to three figure accuracy. The three world lines show that, unlike simple harmonic motion, the period of the anharmonic relativistic motion is not independent of the amplitude. For $\beta_0=0.20$ the world line is very close to the sine wave form of nonrelativistic simple harmonic motion. At $\beta_0=0.90$ the curvature is becoming more concentrated at the turning points. At $\beta_0=0.99$ the world line has become markedly anharmonic, being nearly straight between the turning points. Only in the vicinity of the turning points, where the magnitude of the Hooke's law force is maximum and the velocity is becoming nonrelativistic, is the force effective in changing the velocity.

It is interesting to examine the motion in the ultrarelativistic region where $\gamma_0\gg 1$. In this case $\kappa\rightarrow 1$ and $A_+\rightarrow A_-\rightarrow\sqrt{2\gamma_0}$. Then Eq. (37) becomes

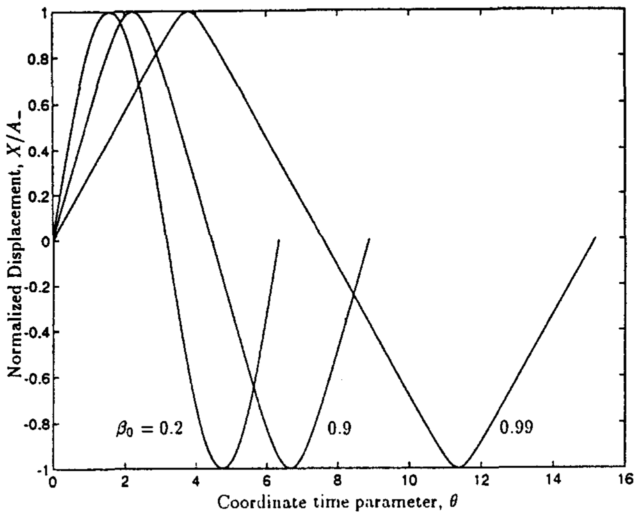

Fig. 2. The normalized displacement $X(\theta)/A_-$ as a function of the coordinate time parameter θ for one complete cycle with $\beta_0=0.20$, 0.90, and 0.99. The corresponding amplitudes are 0.20, 1.61, and 3.49 to three figure accuracy. The world lines become more anharmonic with increasing energy. At $\beta_0=0.99$ the world line is nearly straight between the turning points, indicating that only in these regions where the velocity is becoming nonrelativistic is the force effective in changing the velocity. Unlike simple harmonic motion, the period of the relativistic motion is not independent of the amplitude.

$$\Theta=\sqrt{\frac{2}{\gamma_0}}\int_0^{\phi}\sec\alpha\,d\alpha$$

$$=\frac{1}{\sqrt{2\gamma_0}}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right)=\frac{1}{\sqrt{2\gamma_0}}\ln\left(\frac{\sqrt{2\gamma_0}+X}{\sqrt{2\gamma_0}-X}\right), \quad (42)$$

and Eq. (39) becomes

$$\theta=\gamma_0\Theta-\sqrt{2\gamma_0}\int_0^{\phi}\tan\alpha\sin\alpha\,d\alpha. \quad (43)$$

Substituting Eq. (42) for Θ in Eq. (43) and integrating by parts, after a cancellation we obtain the simple result,

$$\theta=\sqrt{2\gamma_0}\sin\phi=X. \quad (44)$$

In the ultrarelativistic region the world line approaches that of a photon, $x=ct$, and the effect of the force is negligible. The effective kinetic energy at the origin just equals the maximum effective potential energy, $T_0=\frac{1}{2}\gamma_0=V_{\max}$, and the amplitude becomes very large: $A=(X)_{V_{\max}}=\sqrt{2\gamma_0}\gg 1$. In the limiting case where $\gamma_0\rightarrow\infty$, the amplitude $A\rightarrow\infty$, and the motion ceases to be periodic. Of course a massive particle can never reach this limit.

IV. CONCLUSION

We have presented a relativistic generalization of the motion of a one-dimensional simple harmonic oscillator. Simple harmonic motion is a key concept in physics, and it is of intrinsic interest to extend the model into the relativistic domain. We have derived the coordinate-time and the proper-time equations of motion in the Lagrangian formalism and solved the latter analytically in terms of standard functions, finally obtaining the world line $t(x)$.

We have analyzed the proper-time relativistic motion in terms of an effective potential energy in an analogy with classical mechanics. There is a different effective potential energy for different values of total relativistic energy. A massive particle never has enough effective energy to surmount the maximum of the effective potential barrier, and the motion is bounded and periodic for all $\beta_0<1$. As β_0 is increased from zero toward one, the world line of the relativistic oscillator changes from the sine wave of simple harmonic motion to an anharmonic periodic wave with the curvature concentrated more and more at the turning points and the period increasing with amplitude. The anharmonicity is a relativistic effect due to time dilation along the world line.

As this problem is solvable in terms of standard functions, it would serve as a useful and interesting extension to the usual treatment of hyperbolic motion in a special relativity course. Among other things, it is an excellent illustrative example of the effectiveness of a force in altering the velocity in special relativity.

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[7]Ray Skinner, *Relativity* (Blaisdell, Waltham, MA, 1969), p. 208, prob. 2.73.
[8]Noel A. Doughty, *Lagrangian Interaction* (Addison–Wesley, Reading, MA, 1990), pp. 308–310.
[9]See Ref. 4, p. 87.
[10]See Ref. 1, pp. 61–62, 343.
[11]H. C. Ohanian, *Gravitation and Spacetime* (Norton, New York, 1976), p. 304, prob. 9.
[12]I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products*, corrected and enlarged edition (Academic, San Diego, 1980).

THE EYES OF AN EDITOR

What scientists *do* has never been the subject of a scientific, that is, an ethological enquiry. It is no use looking to scientific 'papers', for they not merely conceal but actively misrepresent the reasoning that goes into the work they describe. If scientific papers are to be accepted for publication, they must be written in the inductive style. The spirit of John Stuart Mill glares out of the eyes of every editor of a Learned Journal.

Peter Medawar, *Pluto's Republic* (Oxford, New York, 1984), pp. 132–133.