the string tension T_0 in the string supporting pulley no. 0. Show that Eq. (2) becomes $a_{20} = [(m_1 - m_2)/4m_1m_2]T_0$; $a_{2i} = a_{20} + a_{0i}$, and that the remaining equations are unchanged.

(3) Find the tensions in all of the strings. For example, for Fig. 3, $T_2 = m_2(g + a_{2i})$, where m_2 is given by Eqs. (4) and (6), and a_{2i} by Eq. (2).

(4) Generate and solve other compound machines. For example, start with Fig. 2 and replace m_1 by a machine with masses m_a and m_b. Do the same for Fig. 3. Then perhaps replace m_3 by a machine with masses m_c and m_d, for either Fig. 2 or Fig. 3. You will find that the problem always "remains diagonal." The solutions can be simply written down.

(5) Generalize to more realistic pulleys. In Fig. 1 let the frictionless pulley have nonzero mass m_0, moment of inertia I_0, and radius R_0. (To prevent the massless string connecting m_1 and m_2 from slipping we may replace it by a massless flexible chain whose links mesh with teeth on the circumference of the pulley.) Show that Eq. (2) becomes generalized to

$$a_{20} = [(m_1 - m_2)/(m_1 + m_2 + I_0/R_0^2)](g + a_{0i}),$$
$$a_{2i} = a_{20} + a_{0i}, \qquad (2')$$

while Eq. (3) for the effective mass (now called M_0) is generalized to

$$M_0 = m_0 + [4m_1m_2 + (m_1 + m_2)I_0/R_0^2]/[m_1 + m_2 + I_0/R_0^2]. \qquad (3')$$

(We designate the effective mass by M_0 rather than m_0, because there is now a real mass m_0 to be included.) (Hint: Note that because I_0 is not zero we no longer have tension $T_1 = T_2$; and because m_0 is not zero we no longer have $T_0 = T_1 + T_0$.) Now go to the compound machine of Fig. 2, but let pulley no. 2 have real mass m_2, moment of inertia I_2 and radius R_2. By analogy with Eqs. (2′) and (3′) find the generalizations of Eqs. (4) and (5), calling M_2 the effective mass of pulley no. 2. You can now go to any multiply compound machine. For example, go to Fig. 3 and make the corresponding generalizations of Eqs. (6), (7), and (8) when all pulleys have nonzero m,I, and R.

[1]The present article was inspired by solving Problem 14, Chap. 6, in *Physics*, by H. C. Ohanian (Norton, New York, 1985). That problem has three masses and two pulleys, as in our Fig. 2. (Yes, I got the same answer that they did!)

Relativistic telemetry

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Many relativistic formulas can be derived without explicitly using Lorentz transformations but, instead, directly from Einstein's postulate: "All physical laws are the same for all inertial observers." These formulas include time dilation, addition of velocities, the Doppler effect, and optical aberration. From the visual picture seen by one observer, one can deduce the picture seen at the same space-time point by any other observer, without knowing anything about the speed or the three-dimensional shape of the observed object. In particular, the apparent visual shape of a moving object is not contracted, but rotated.

I. INTRODUCTION

The Lorentz transformation is the standard way to derive formulas for relativistic phenomena, such as time dilation, addition of velocities, the Doppler effect, optical aberration, etc. Although the derivation of these formulas is straightforward, it is rather formal and not very transparent from the point of view of physics. In this article, I show how they can be derived very simply from Einstein's relativity principle: *All physical laws are the same for all inertial observers*. (In particular, the speed of light is the same.)

Sections II, III, and IV of this article refer to problems in one, two, and three space dimensions, respectively. A similar approach has been proposed by other authors[1,2] for 1-dimensional problems. Its extension to 2- and 3-dimensional problems is apparently new.

II. ONE SPACE DIMENSION

In this section, I derive the formulas for time dilation, the collinear Doppler effect, and the addition of collinear velocities. Consider two inertial observers, A and B, receding from each other with relative velocity v. Two light signals are sent by A toward B at times t_0 and $t_0 + \Delta t_0$. They are reflected by B (who holds a mirror) and return to A at times t_2 and $t_2 + \Delta t_2$. The calculation is easiest in the frame in which A is at rest (see Fig. 1). When the first signal is reflected by B, the latter is passing in front of another inertial observer A′, who is at rest with respect to A. The clock of A′, synchronized with that of A, shows time t_1. Likewise, when B reflects the second signal, he is in front of yet another observer, A″, at rest with respect to A and synchronized with A, and the clock of A″ then shows $t_1 + \Delta t_1$. An

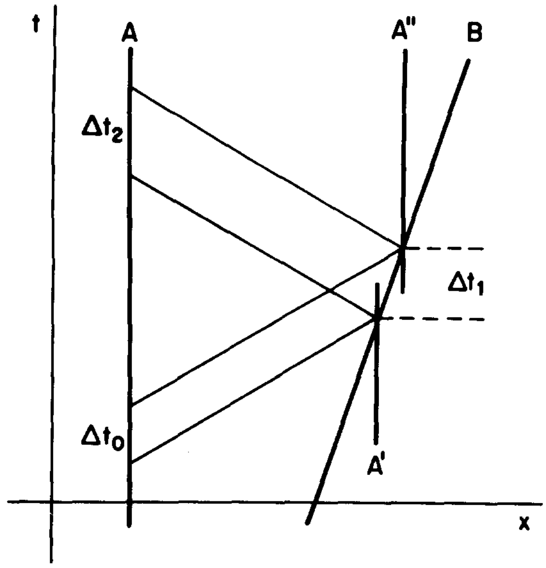

Fig. 1. The double Doppler effect, as described in the frame where A, A′, and A″ are at rest, and B moves with uniform velocity with respect to them.

elementary *nonrelativistic* calculation gives

$$\Delta t_1 = [c/(c-v)]\Delta t_0 \tag{1}$$

and

$$\Delta t_2 = [(c+v)/c]\Delta t_1, \tag{2}$$

whence

$$\Delta t_2 = [(c+v)/(c-v)]\Delta t_0. \tag{3}$$

Notice that Eq. (3) refers to events observed by A and is independent of any synchronization convention. No relativity principle was used until now. Equation (3) is also valid for acoustic signals, c being the velocity of sound, provided that A is at rest in the acoustic medium (i.e., the velocity of sound with respect to A is the same in both directions).

Now suppose that B holds not only a mirror but also a clock, built identically to the clock of A. What is the time interval $\Delta t_1'$ observed by B? (Notice that B's clock cannot be synchronized with those of A, A′, and A″, because it moves with respect to them.) Obviously, we must have a relationship of the type

$$\Delta t_1' = f(v)\Delta t_0, \tag{4}$$

where $f(v)$ is the *one-way Doppler factor* relating the time intervals *measured by the receiver and the emitter*, respectively. This factor is some function of the relative velocity v, to be determined. Likewise, for the reflected signals,

$$\Delta t_2 = f(v)\Delta t_1'. \tag{5}$$

Now comes the crux of the argument: The Doppler factors in (4) and (5) must be the same, because *they describe the same phenomenon*—namely the effect of the relative velocity v on the time intervals between a pair of light signals, as measured by the emitter (right-hand side) and the receiver (left-hand side). It is here that we explicitly assume that the speed of *light* (contrary to that of sound waves, say) is the same for all inertial observers.

It follows from (4) and (5) that $\Delta t_2 = [f(v)]^2\Delta t_0$ and

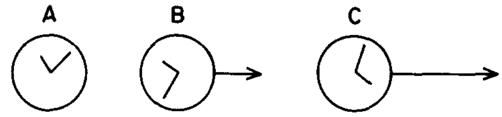

Fig. 2. Observer B moves with velocity v with respect to A. Observer C moves with velocity u with respect to B, and velocity w with respect to A.

therefore, from Eq. (3),

$$f(v) = [(c+v)/(c-v)]^{1/2}. \tag{6}$$

Comparing (1), (4), and (6), one obtains

$$\Delta t_1' = \Delta t_1(1 - v^2/c^2)^{1/2}, \tag{7}$$

the familiar time dilation formula.

Now consider a third observer, C, moving with velocity u with respect to B, and velocity w with respect to A (see Fig. 2). What is the relationship between u, v, and w? Let A send a pair of signals toward B and C. The time interval measured by A is Δt_0. That measured by B is

$$\Delta t_1' = [(c+v)/(c-v)]^{1/2}\Delta t_0. \tag{8}$$

These signals then continue toward C who observes them with a time interval

$$\Delta t_2'' = [(c+u)/(c-u)]^{1/2}\Delta t_1' \tag{9}$$

or

$$\Delta t_2'' = [(c+w)/(c-w)]^{1/2}\Delta t_0. \tag{10}$$

Consistency implies that

$$\frac{c+w}{c-w} = \frac{c+u}{c-u}\,\frac{c+v}{c-v}, \tag{11}$$

whence

$$w = \frac{u+v}{1+uv/c^2}, \tag{12}$$

the familiar law of addition of velocities.

III. TWO SPACE DIMENSIONS

This section treats, by the same direct methods, the noncollinear Doppler effect and the optical aberration formula. I start with the latter.

To give a concrete example, consider a radar and a missile, as in Fig. 3. The problem is to find the relationship between the angle θ, measured in the frame where the radar is at rest, and the angles θ' and θ'', measured in the frame where the missile is at rest. The relationship between θ' and θ'' is readily obtained form Fig. 3(b):

$$v(t' + t'') = ct''\cos\theta'' - ct'\cos\theta', \tag{13}$$

$$ct'\sin\theta' = ct''\sin\theta''. \tag{14}$$

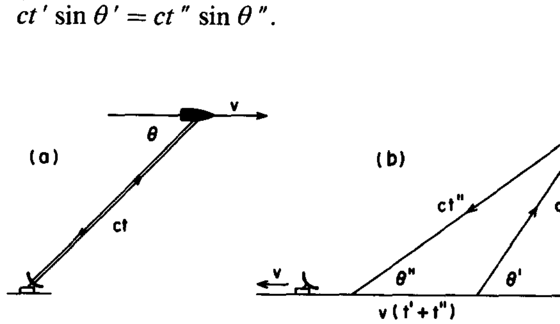

Fig. 3. A radar sends a signal toward a missile and receives the echo. (a) Description in the frame where the radar is at rest. (b) Description in the frame where the missile is at rest.

Notice that these two equations are obtained in the *same* inertial frame and again there is no problem of clock synchronization. Nevertheless, relativity is implicit in the assumption that the velocity of light is the same in both directions.

Dividing (13) by (14), one obtains

$$\frac{v}{c}\left(\frac{1}{\sin\theta'}+\frac{1}{\sin\theta''}\right)=\cot\theta''-\cot\theta', \tag{15}$$

whence, by elementary trigonometry,

$$\frac{\tan(\theta''/2)}{\tan(\theta'/2)}=\frac{c-v}{c+v}. \tag{16}$$

This is the well-known law of reflection from a moving mirror.[3]

We now turn our attention to θ, as observed by the radar. First consider the signal from the radar to the missile. The angles θ and θ' must be related by an equation such as

$$\theta=F(\theta',v), \tag{17}$$

where the function F has to be determined. Notice that the angle in the left-hand side (lhs) of (17) is the one observed by the emitter, and the angle in the right-hand side (rhs) the one observed by the receiver.

Likewise, for the reflected ray, we must have

$$\theta''=F(\theta,v), \tag{18}$$

with the *same function F* as in (17), because this is a *description of the same phenomenon*: Here again, the emitter angle is in the lhs, and the receiver angle in the rhs. Comparison with (16) then gives the well-known relativistic aberration formula[4]

$$\frac{\tan(\theta''/2)}{\tan(\theta/2)}=\frac{\tan(\theta/2)}{\tan(\theta'/2)}=\left(\frac{c-v}{c+v}\right)^{1/2}. \tag{19}$$

This result is the unique solution of $\theta''=F[F(\theta',v),v]$ as can easily be seen if we replace θ' by $\tan(\theta'/2)$, etc. Notice the similarity of (19) with Eq. (8). It implies that if a distant star is observed by several telescopes moving with respect to each other (or by a single telescope on the Earth, orbiting around the Sun[5]) then a comparison of the different aberration angles will enable us to infer the *relative* velocities of the telescopes, but not their "absolute" velocity with respect to the star.

Equation (19) can also be derived by considering the noncollinear Doppler effect (see Fig. 4). We have $c(t_1-t_0)=r(t_1)$. Differentiation with respect to t_1 gives

$$c\left(1-\frac{dt_0}{dt_1}\right)=\frac{dr(t_1)}{dt_1}=\frac{\mathbf{r}\cdot\mathbf{v}}{r}=v\cos\theta, \tag{20}$$

whence

$$\frac{dt_1}{dt_0}=\frac{c}{c-v\cos\theta}, \tag{21}$$

which is the generalization of (1). Likewise, differentiation of $c(t_2-t_1)=r(t_1)$ gives

$$\frac{dt_2}{dt_1}=\frac{c+v\cos\theta}{c} \tag{22}$$

which is the generalization of (2). It follows that, in the frame where the missile is at rest, we have

$$\frac{dt'_1}{dt'_0}=\frac{c+v\cos\theta'}{c}. \tag{23}$$

Indeed, (22) and (23) must have *the same form*, because

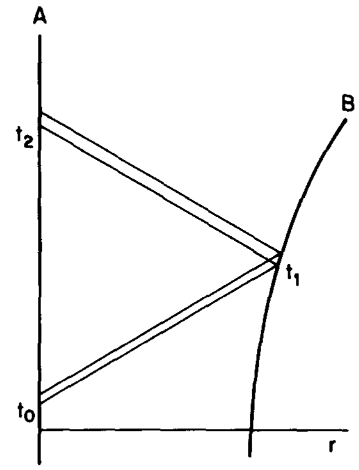

Fig. 4. The general Doppler effect: A pair of signals is sent by A (at rest) and reflected by B, whose distance r from A is an arbitrary function of time.

both apply to the *same physical situation*, namely the receiver being at rest. In both cases, the angles θ and θ' are those between the emitter velocity and the signal velocity [the signal is the reflected ray in Fig. 3(a) and the emitted ray in Fig. 3(b), respectively].

The difference between (21) and (23) is due to the fact that (21) is computed in the frame where the radar is at rest, while (23) applies to the frame where the missile is at rest. The one-way Doppler factor, which relates time intervals *as measured by the emitter and the receiver themselves*, is

$$\frac{dt'_1}{dt_0}=(1-\beta^2)^{1/2}\frac{dt_1}{dt_0}=(1-\beta^2)^{-1/2}\frac{dt'_1}{dt'_0}, \tag{24}$$

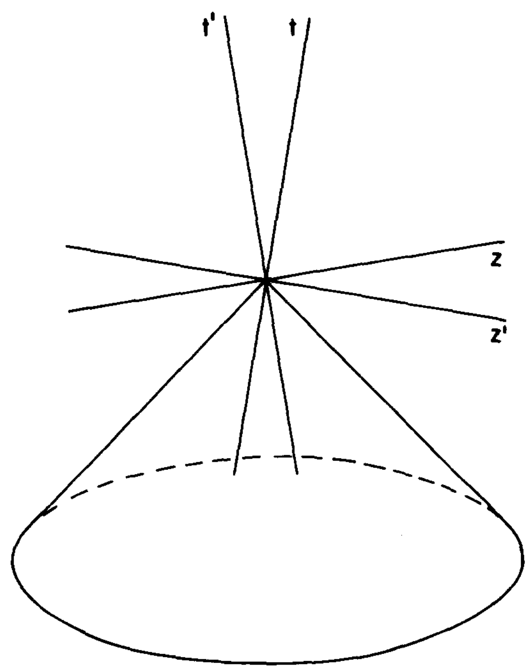

Fig. 5. The common past light cone of two observers in uniform relative motion. The t and t' axes are the world lines of the observers. The z and z' axes are parallel to their relative velocity. The $x=x'$ and $y=y'$ axes are perpendicular to the plane of the paper.

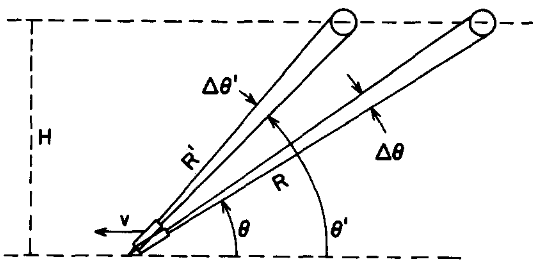

Fig. 6. The apparent distance to a small sphere of known diameter D is evaluated as $R = D/\Delta\theta$ and $R' = D/\Delta\theta'$, respectively, by two observers with relative velocity v. Then $H = R\sin\theta = R'\sin\theta'$ appears to be the same for both observers.

because of the relativistic time dilation. Here, $\beta = v/c$, as usual. We thus obtain[2]

$$\frac{dt'_1}{dt_0} = \frac{(1-\beta^2)^{1/2}}{1-\beta\cos\theta} = \frac{1+\beta\cos\theta'}{(1-\beta^2)^{1/2}}. \tag{25}$$

Consistency implies that

$$\cos\theta - \cos\theta' = \beta(1-\cos\theta\cos\theta'), \tag{26}$$

which is yet another proof of the relativistic aberration formula, Eq. (19). With the help of some additional trigonometry, the one-way Doppler factor can also be written as[4]

$$\frac{dt'_1}{dt_0} = \frac{\sin\theta'}{\sin\theta} = \frac{d\theta'}{d\theta}. \tag{27}$$

IV. THREE SPACE DIMENSIONS

It has been known for a long time[6–8] that the Lorentz contraction of a moving object is "invisible" because it is compensated by different retardations of the signals originating in different part of the object. Thus, if a snapshot is taken of a moving object, the latter does *not* appear contracted, but rather *rotated*. This statement is actually valid only in the limiting case of objects which are very small or very distant. Those subtending a *finite* solid angle are conformally distorted. However, it may be shown that a spherical object of any size, moving at any speed, always appears to have a circular boundary.[9]

The proof of these statements for a small, distant object is very simple. What an observer actually sees is his *past light cone* (Fig. 5). The visual image, or the photographic record, can be expressed by a pair of angles, such as a polar angle θ measured from some arbitrary direction, and an azimuthal angle ϕ measured around that direction. When two observers in relative motion open the shutters of their cameras at the same space-time point, they see the same things: Both cameras collect photons from the same past events (the past light cone is Lorentz invariant). The only difference in the photographic records is due to aberration. The polar angles θ and θ' (which it is convenient to measure from the direction of the relative velocity **v**) are different, and are related by Eq. (19). The azimuthal angles ϕ and ϕ' are obviously equal, by rotational symmetry around **v**.

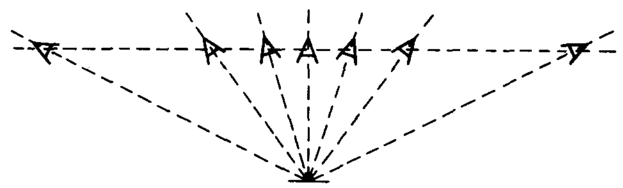

Fig. 7. An object (the letter A) seen by seven observers at the same point and the same instant. One observer is at rest with respect to the object, and just beneath it. The six others move with velocities $\pm 0.3c$, $\pm 0.6c$, and $\pm 0.9c$.

The *angular magnification* in the polar direction is $d\theta'/d\theta$. That in the azimuthal direction is $\sin\theta'\,d\phi'/\sin\theta d\phi$. These two magnifications are equal, by virtue of Eq. (27) and of $\phi' = \phi$. Therefore the visual shape of a small object is magnified, without distortion, by the same factor. More formally, the angular distance $d\alpha$ of two neighboring rays, given by

$$d\alpha^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{28}$$

transforms as

$$\frac{d\alpha'}{d\alpha} = \frac{\sin\theta'}{\sin\theta}, \tag{29}$$

independently of the ratio $d\theta/d\phi$.

This has an interesting consequence. Suppose that the true size of the object is known and that one uses the visual angle $d\alpha$ to determine the distance to that object, as in Fig. 6. Then if we draw through the object a line parallel to the relative velocity **v** of a pair of observers, the latter will *agree* about their distance H to that line.

In summary, all the observers see the *same image* (the same apparent shape) but each one sees the object in a different direction, because of the aberration. The object thus appears to them *rotated* by the aberration angle $\theta' - \theta$. This is illustrated in Fig. 7, for several observers moving with collinear relative velocities.

ACKNOWLEDGMENTS

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