In search of the "starbow": The appearance of the starfield from a relativistic spaceship

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One prediction for the appearance of the starfield from a moving reference frame has been circulated widely despite physically objectionable features. We reexamine the physical basis for this effect. To the well-known aberation and Doppler effects we add the transformation of intensity of light upon a change of reference frame. We integrate the transformed spectrum of a star, represented as a blackbody, over the response function of the human eye. We present a one-parameter function to represent the dependence of the apparent visual magnitude of different stars upon Doppler factor. We conclude with a sequence of computer-generated figures to show the appearance of Earth's starfield at various velocities. A "starbow" does not appear.

I. INTRODUCTION

The body of literature of science fiction (SF) offers an opportunity for physics educators to interact with persons who may never encounter physics in any other way. The artistry of the SF author can often hold a reader's attention where the best textbook cannot. It is the hope, but seldom the reality, that the physics of a particular SF story is basically correct. However, even a flawed presentation can serve as the springboard to an appreciation of what physics really is, if it leads a reader to question whether the universe really behaves that way. Of course there are some aspects of pseudophysics (such as time machines or faster-than-light travel) which are so widely employed in SF as to become clichés, but for which one can only point out the conflict with physics as we understand it. However, other examples occur in SF which seem much more plausible and which may require a careful investigation to establish their truth or falsehood. It is even more interesting if the investigation turns up relatively straightforward pieces of physics which nevertheless seem to be little circulated or appreciated among professional physicists themselves. For the investigator there is also the challenge of trying to trace the origin of concepts in an activity like SF where one regularly does not cite ones predecessors. This investigation combines all of these features.

The "starbow" of our title is the purported appearance of a typical starfield (such as that seen from earth) when viewed from a spaceship moving relative to the stars at a speed greater than about 0.4 times the speed of light. The word appears in the title of at least one SF story, and of a collection of stories by that author.[1] The concept is sufficiently widespread that an article accompanied by a work of graphic art recently appeared in a popular scientific journal.[2] The supposed explanation of this phenomenon is that any stars close to the forward path of the ship will have their light shifted via the Doppler effect into the ultraviolet and hence will be invisible to the eye. Similarly, any stars close to the backward direction will have been Doppler shifted into the invisible infrared. The stars which remain visible will be Doppler shifted by lesser amounts so as to appear at various colors other than their normal color. Those stars will also appear displaced toward the forward direction, so that the center of the "starbow" is at an acute angle from the forward direction.

A closely related phenomenon of current interest to professional astrophysicists is the anisotropy of the temperature of the cosmic blackbody radiation at ~ 3 K.[3] In that case the anisotropy is caused by the net motion of the earth relative to a preferred reference frame in which the radiation is isotropic.

In Sec. II of this paper we specify the reference frames to be employed and present the well known relations for Doppler effect and aberration. In Sec. III we present the much less well known relation for the transformation of the intensity of light as measured in each frame. In Sec. IV we represent a star as a blackbody and give expressions for its spectral intensity in each frame. In Sec. V we integrate this spectrum over a reasonable response function for the human eye in order to evaluate the perceived brightness. We present a one-parameter expression which approximates the result of the numerical integration. In Sec. VI we display a sequence of computer-generated figures to show the appearance of earth's starfield as viewed in reference frames moving toward the north celestial pole at various speeds, incorporating the perceived brightness of the stars. In Sec. VII we discuss the results of our model and compare it to earlier models, particularly to the model which predicts the "starbow." In an appendix we present a model for the intensity transformation based on counting photons in each reference frame.

II. ABERRATION AND DOPPLER EFFECT

Let us define the reference frames in which observations are to be compared. Frame S ("rest frame") is determined by the average motion of the local stars, within perhaps a few hundred parsecs. Typical stars will have quite small velocities in this frame, on the order of $10^{-4}c$. For the purposes of this investigation, they can be considered as fixed in frame S. Frame S' ("ship frame") is determined by the spaceship which moves at variable velocity $\mathbf{v}$ relative to frame S. We shall normally take $\mathbf{v}$ as fixed in direction, defining the z and z' axes. With the x and x' axes chosen parallel, the usual equations of the Lorentz transformation apply:

$$cdt' = \gamma(cdt - \beta dz);$$
$$dx' = dx; \qquad dy' = dy;$$
$$dz' = \gamma(dz - \beta cdt). \tag{1}$$

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We employ the usual notation with $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$.

The observations consist of measurements of the light from the stars of the field. The stars are typically so distant that their light can be considered as plane waves or as photons propagating parallel. We specify the location of a star from the event point of observation by spherical polar angles in the rest frame. It is at colatitude angle θ from the line of motion of the ship and at azimuth angle ϕ about the line of motion, measured from the arbitrary x axis. The propagation four vector of a monochromatic component of the starlight is thus

$$(k;\mathbf{k}) = (\omega/c;\, k_x,k_y,k_z) = (k;\, -k\sin\theta\cos\phi,\, -k\sin\theta\sin\phi,\, -k\cos\theta). \quad (2)$$

[The energy-momentum four vector of the corresponding photons is $(\mathscr{E}/c;\, \mathbf{p}) = \hbar(k;\mathbf{k}) = (\hbar\omega/c;\hbar\mathbf{k})$.] To find the propagation four vector in the ship frame we apply the Lorentz transformation [Eq. (1)]:

$$k' = \gamma(k - \beta k_z) \quad \text{or} \quad k' = \gamma(k + \beta k\cos\theta); \quad (3a)$$

$$k'_x = k_x \quad \text{or} \quad -k'\sin\theta'\cos\phi' = -k\sin\theta\cos\phi; \quad (3b)$$

$$k'_y = k_y \quad \text{or} \quad -k'\sin\theta'\sin\phi' = -k\sin\theta\sin\phi; \quad (3c)$$

$$k'_z = \gamma(k_z - \beta k) \quad \text{or} \quad -k'\cos\theta' = \gamma(-k\cos\theta - \beta k). \quad (3d)$$

From Eq. (3a) we identify the angle-dependent Doppler effect: $k' = Dk$ or $\omega' = D\omega$ or $\lambda' = \lambda/D$, where

$$D = D(\theta) = \gamma(1 + \beta\cos\theta). \quad (4)$$

From the ratio of Eqs. (3a) and (3d) we identify one expression for aberration

$$\cos\theta' = (\cos\theta + \beta)/(1 + \beta\cos\theta). \quad (5)$$

From the ratio of Eqs. (3b) and (3c) we find the invariance of the azimuth angle

$$\phi' = \phi. \quad (6)$$

The ratio of Eqs. (3a) and (3b) yields an alternative expression for aberration

$$\sin\theta' = \sin\theta/D. \quad (7)$$

The inverse to the Lorentz transformation yields additional useful forms

$$\cos\theta = (\cos\theta' - \beta)/(1 - \beta\cos\theta'), \quad (8)$$

$$D = \gamma^{-1}(1 - \beta\cos\theta')^{-1}. \quad (9)$$

For a given star, Eqs. (4) and (9) give the same value for the Doppler factor.

All of these expressions are perfectly well known and appear in essentially every exposition on relativity since Einstein first found them.[4] Their immediate effect on the appearance of the starfield is also widely reported.[5-8] For any star not exactly on the line of motion, aberration causes the star image viewed in the ship frame to be displaced forward while keeping the same azimuth. The extent of the forward displacement increases as v increases. For the idealized case of a uniform distribution of stars in the rest frame, one would see in the ship frame that half the star images are crowded into the cone of half angle $\theta' = \cos^{-1}\beta$ about the forward direction.

One effect of the Doppler factor is also clear. Stars generally forward have $D > 1$, which means that their light is shifted to shorter wavelengths ("blue shifted") as viewed in the ship frame. The maximum shift occurs directly forward and is given by the factor $D = D(0) = [(1 + \beta)/(1 - \beta)]^{1/2}$. Stars generally backward have $D < 1$, and their light is shifted to longer wavelengths ("red shifted"). The extreme shift occurs directly backward and is given by the factor $D = D(\pi) = [(1 - \beta)/(1 + \beta)]^{1/2}$. Some stars experience no Doppler shift at all, namely those for which $D = 1$. These stars are at colatitude angle $\theta = \cos^{-1}[(1 - \gamma)/\gamma\beta]$, and aberration shifts their images in the ship frame to the supplementary angle $\theta' = \cos^{-1}[(\gamma - 1)/\gamma\beta]$.

III. INTENSITY TRANSFORMATION

When the frequency (or wavelength) of a monochromatic beam of light transforms as $\omega' = D\omega$ (or $\lambda' = \lambda/D$) in going from frame S to frame S', the intensity of the beam (energy per unit area per unit time) transforms as

$$I' = D^2 I. \quad (10)$$

This can be seen in several different ways. In the first method, originated by Einstein[4] and reproduced elsewhere,[9-13] one transforms the electric and magnetic fields of a plane wave. If the amplitude of the wave is designated by E_0, one finds $E'_0 = DE_0$. Then since the intensity is proportional to E_0^2, one finds $I' = D^2 I$. Einstein employed this result to show that the total energy in a finite classical wave packet experiences exactly the same Doppler effect as the frequency. In a second method,[14] one forms the symmetric stress-energy-momentum tensor from the fields of the plane wave and transforms that, for the same result. In a third method, one considers the beam to consist of photons. The energy of each photon transforms as $hc/\lambda' = D(hc/\lambda)$. In addition, the rate at which photons cross a unit area normal to the beam transforms with another factor D, to give the overall intensity transformation of Eq. (10). Since this intensity transformation is so important to our result, and since we consider this third method of counting photons so transparent, we describe it further in the Appendix.

We can also find the transformation for the spectral intensity of a beam with an arbitrary distribution of frequencies or wavelengths. Let the energy be distributed in wavelength so that the quantity $dI = S(\lambda)d\lambda$ is the energy per unit area per unit time with wavelengths between λ and $\lambda + d\lambda$. Upon transformation to frame S', this increment of flux is given by $dI' = D^2 dI$, and it is in the wavelength range λ' to $\lambda' + d\lambda'$ so that we write $dI' = S'(\lambda')d\lambda'$. Combining these relations with $\lambda' = \lambda/D$, we find

$$S'(\lambda') = D^3 S(\lambda). \quad (11)$$

IV. BLACKBODY MODEL FOR STELLAR EMISSION

Let us assume that a star in its proper frame (which is essentially the same as frame S) can be described by a single surface temperature T. Then every unit area of its surface emits energy in the wavelength interval between λ and $\lambda + d\lambda$ during time dt according to the blackbody radiation law[15]

$$F(\lambda,T)d\lambda dt = 2\pi hc^2\lambda^{-5}[\exp(hc/\lambda k_BT) - 1]^{-1}d\lambda dt. \quad (12)$$

(We designate Boltzmann's constant as k_B to distinguish it from wave number.) This assumption is equivalent to ignoring any extended stellar atmosphere, which could increase or decrease the net emission at various frequencies.

If we take the star to have radius a and surface area $4\pi a^2$, then when observed at distance $R \gg a$ in frame S it will produce an intensity per unit wavelength interval given by

$$S(\lambda,T) = (a/R)^2 2\pi hc^2\lambda^{-5}[\exp(hc/\lambda k_BT) - 1]^{-1}. \quad (13)$$

This is equivalent to ignoring any scattering or absorption in the intervening interstellar medium. For visible light the major agent for absorption is dust, but ultraviolet light of wavelength shorter than the Lyman limit (912 Å) will be absorbed by the ubiquitous neutral hydrogen of interstellar space.

We now use Equation 11 to transform the observation to frame S', where $\lambda' = \lambda/D$:

$$\begin{aligned} S'(\lambda',T) &= D^3(a/R)^2 2\pi hc^2(\lambda'D)^{-5} \\ &\quad \times [\exp(hc/\lambda'Dk_BT) - 1]^{-1} \\ &= D^{-2}(a/R)^2 2\pi hc^2\lambda'^{-5} \\ &\quad \times [\exp(hc/\lambda'k_BDT) - 1]^{-1}. \quad (14) \end{aligned}$$

Apart from the factor D^{-2}, this is exactly the spectrum of a blackbody at the Doppler-shifted temperature DT. If one uses a (hypothetical) detector which is uniformly responsive to all wavelengths, one finds for the integrated intensity

$$\begin{aligned} I' &= \int_0^\infty S'(\lambda',T)d\lambda' \\ &= D^{-2}\left(\frac{a}{R}\right)^2 2\pi hc^2 \int_0^\infty \lambda'^{-5}\left[\exp\left(\frac{hc}{\lambda' k_BDT}\right) - 1\right]^{-1} d\lambda' \\ &= D^2\left(\frac{a}{R}\right)^2 2\pi hc^2 \int_0^\infty (\lambda'D)^{-5} \\ &\quad \times \left[\exp\left(\frac{hc}{\lambda'Dk_BT}\right) - 1\right]^{-1} d(\lambda'D) \\ &= D^2 I. \end{aligned}$$

This last expression merely verifies again the relation for transformation of intensity.

V. BRIGHTNESS PERCEIVED BY THE HUMAN EYE

It is more interesting to consider the human eye as the detector because it is sensitive to color and because its range of response over wavelength is quite limited. Color can be discriminated only for stars which are bright enough to stimulate cone vision,[16] since the rod vision of the fully dark-adapted eye is effectively color blind. The color sequence runs from red for stars with $T \approx 2500$ K through orange, yellow, and white, becoming blue white for stars with $T \gtrsim 20\,000$ K. This color perception depends essentially upon the slope of the star's spectrum within the range of the eye response, as seen in Fig. 1. In the ship frame, T is transformed to DT, so that there will be an excess of bluer stars at forward angles where $D > 1$, and an excess of redder

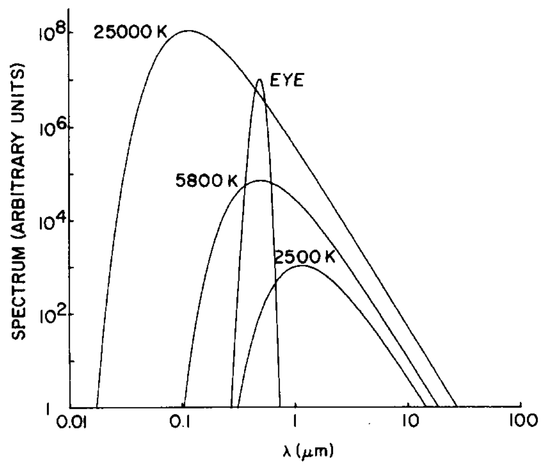

Fig. 1. Blackbody spectra and human eye response. A hot star (25 000 K) has relatively more radiation of short wavelength within the visible range and appears blue white. A solar-type star (5800 K) has relatively uniform intensity across the visible range and appears yellow. A cool star (2500 K) has relatively more radiation of long wavelength and appears red.

stars at backward angles where $D < 1$. Even in the forward direction not all objects will be blue white, since an infrared object such as a protostar embedded in a dust cloud may have its Doppler-shifted temperature appropriate for orange or yellow.

We have chosen to represent the response function of the human eye to equal intensities of light of different monochromatic colors by a Gaussian function in wavelength

$$f(\lambda) = A\exp[-(\lambda - \lambda_0)^2/\epsilon^2]. \quad (15)$$

With $\lambda_0 = 500$ nm and $\epsilon = 57$ nm this is a reasonable fit for the dark-adapted eye. For the eye adapted to bright light (cone vision), the central wavelength λ_0 is about 560 nm. The perceived brightness in the ship frame is the integral of the spectrum [Eq. (14)] over this response function

$$\begin{aligned} W(D,T) &= \int_0^\infty S'(\lambda')f(\lambda')d\lambda' \\ &= D^{-2}\left(\frac{a}{R}\right)^2 2\pi hc^2 A \\ &\quad \times \int_0^\infty \frac{\lambda'^{-5}\exp[-(\lambda' - \lambda_0)^2/\epsilon^2]d\lambda'}{\exp(hc/\lambda' k_BDT) - 1}. \quad (16) \end{aligned}$$

Figure 2 shows the spectrum and the eye response for several values of D at a single T. To evaluate this integral let us introduce the dimensionless variable $x = (\lambda' - \lambda_0)/\epsilon$ or $\lambda' = \lambda_0 + \epsilon x$:

$$W(D,T) = D^{-2}(a/R)^2 2\pi hc^2\epsilon A\lambda_0^{-5}B(DT). \quad (17)$$

In the integral expression for $B(DT)$ the lower limit is so far from the center of the Gaussian ($\lambda_0/\epsilon = 8.77$) that it can be replaced by $-\infty$ with negligible error. With the numerical values inserted we have

$$\begin{aligned} B(DT) &= \int_{-\infty}^{\infty} (1 + 0.114x)^{-5}e^{-x^2}dx \\ &\times \{\exp[28\,776\text{ K }(DT)^{-1}(1 + 0.114x)^{-1}] - 1\}^{-1}. \quad (18) \end{aligned}$$

We have evaluated this integral numerically using Gauss-Hermite integration.[17] For any value of DT greater than

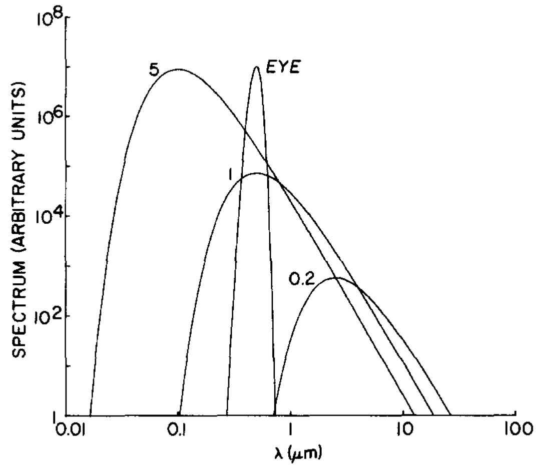

Fig. 2. Doppler shifted blackbody spectra and human eye response. The curve labeled 1 represents the sun (5800 K) viewed at rest where it appears yellow. The curve labeled 5 represents the sun as viewed from a spaceship approaching at speed $\beta = 0.923$ (so that $D = 5$). The sun appears blue white and somewhat brighter than at rest. The curve labeled 0.2 represents the sun as viewed from a spaceship receding at $\beta = 0.923$. The sun appears red but very dim.

1000 K, an evaluation with only five mesh points appears to be valid in that increasing the number of mesh points always gives the same value. The result of the integration is shown as the points in Fig. 3, along with a curve which represents a least-squares fit for values of DT between 10^3 and 10^6 K. The equation of the curve is

$$\log_{10}B = \log_{10}(DT/\mathrm{K}) - 10\,400\ \mathrm{K}/DT + \mathrm{const.} \qquad (19)$$

The greatest difference between this expression and the numerical integration is 0.2 in $\log_{10}B$, or equivalently 0.5 in relative magnitude. Since this is comparable to the limit of brightness discrimination of typical observers, we did not attempt to find a more accurate expression.

We can use this expression in several ways. After substituting Eq. (19) into (17), the perceived brightness in the ship frame can be written

$$\begin{aligned}\log_{10}W(D,T) &= \log_{10}D^{-2} + \log_{10}(DT/\mathrm{K}) - 10\,400\ \mathrm{K}/DT + \mathrm{const}\\ &= -\log_{10}D + \log_{10}(T/\mathrm{K}) - 10\,400\ \mathrm{K}/DT + \mathrm{const.}\end{aligned} \qquad (20)$$

This expression is graphed in Fig. 4 as a function of D for several values of T. This represents the apparent visual magnitude of bodies of one size at one distance, but having different temperature and viewed at various radial veloci-

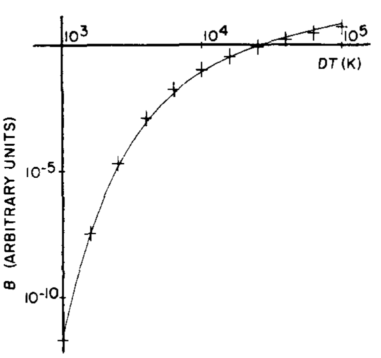

Fig. 3. Comparison of one-parameter fit to integrated perceived brightness. The points represent a numerical integration of Eq. (18), whereas the curve represents Eq. (19).

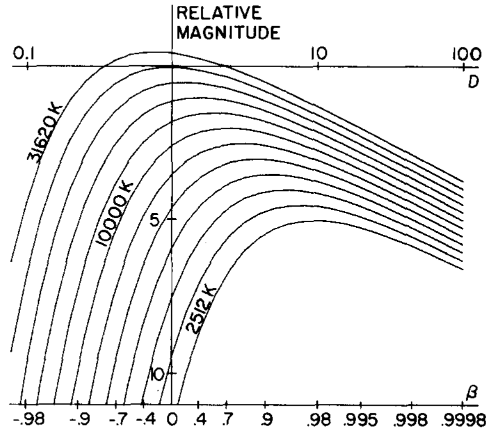

Fig. 4. Relative visual magnitude of blackbodies as a function of temperature and Doppler factor. Each curve represents a single value of temperature in the rest frame. Successive temperatures are at equal steps in $\log_{10}T$.

ties. For actual stars, which are of different sizes and at different distances, it is more appropriate to find the change in brightness caused by the Doppler shift. We have

$$\log_{10}W(D,T) - \log_{10}W(1,T) = 10\,400\ \mathrm{K}(1/T - 1/DT) - \log_{10}D.$$

To express this in magnitudes, multiply by -2.5:

$$m' - m = 2.5\log_{10}D - 26\,000\ \mathrm{K}\,(1/T - 1/DT). \qquad (21)$$

VI. PLOTS OF STARFIELDS

We now combine the results of Sec. II–IV. The stars which we shall plot are the 280 brightest as seen from earth.[18] We have chosen to direct the relative velocity toward the north celestial pole. We plot the aberrated positions of the star images by projecting the forward and backward hemispheres in the ship frame onto two unit circles. We use the radial coordinate $\rho = (1 - \cos\theta')^{1/2}$ in the forward projection and $\rho = (1 + \cos\theta')^{1/2}$ in the backward projection. This has the advantage that equal solid angles of the starfield in the ship frame are projected onto equal areas. Thus the density of star images in area in the projection is proportional to the density in solid angle in the celestial sphere seen from the ship. We indicate the perceived brightness of each star image by the size of the disk representing it. Equal steps in magnitude are represented by equal changes in the area of the disk, and stars which become dimmer than fifth magnitude are not plotted at all. Figures 5–9 show earth's starfield as viewed at successively higher speeds. In Fig. 8 the backward hemisphere is omitted because so few stars remain visible there at that high speed. Figure 9 shows only the central portion of the forward hemisphere at 6× magnification.[19]

VII. DISCUSSION

The effect of aberration shown in Figs. 5–9 is the same as that found by many other authors,[5–8] but the predicted brightness of individual stars is quite different from any previous work. In particular there is no prominent starbow. The reason for this is partially that we have better treated

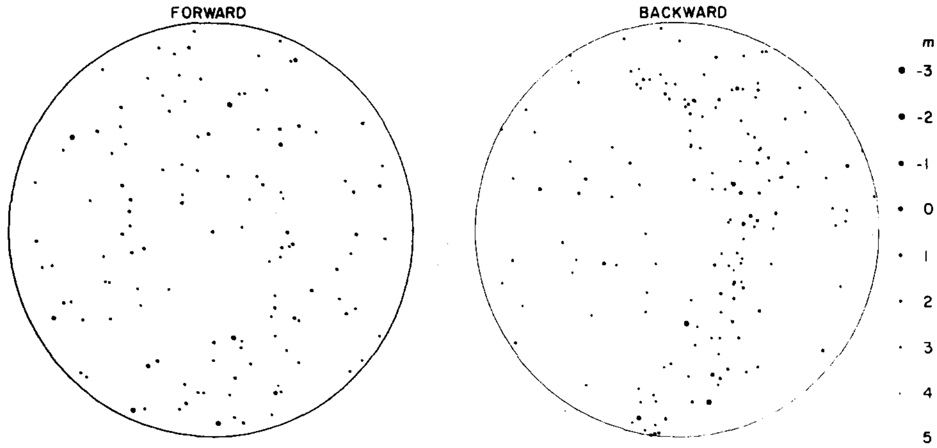

Fig. 5. Earth's starfield viewed at rest. The north celestial pole is at the center of the FORWARD projection. Each hemisphere is plotted so that equal areas represent equal solid angles. The disks to represent different visual magnitudes are used throughout the sequence of Figs. 5–9.

the continuous nature of both the stellar spectrum and the human eye response. More important is the inclusion of the transformation of light intensity, for the first time in such an application.

Our search of the literature seeking coverage of this topic leads us to nominate the intensity transformation as "the most neglected phenomenon of special relativity." It does appear in several monographs,[9–14] but typically with no application beyond that which Einstein devised.[4] It does not appear in a similar number of comparable monographs.[20] In particular it does not appear in the two texts[21,22] which are widely used in the training of U.S. physicists in relativistic electrodynamics, not even as an exercise. Of the many works on aberration and the visual appearance of rapidly moving objects,[8,23,24] we found that only Weisskopf[24] suggested that the intensity is different in the two frames. Unfortunately, his expression is inapplicable and wrong. Scott and van Driel[8] mentioned "the interesting effects of intensity and color change" but they did not pursue the investigation. It is noteworthy that the analysis of the anisotropy of the cosmic blackbody radiation of $\sim$3 K has been done correctly.[25,26]

We have traced the starbow concept back to its apparent origination by Sänger.[5] Since then the concept has circulated[2,27,28] near the fringes of professional science without effective criticism. Sänger assumed "for simplicity's sake" that all stars emit monochromatic light of wavelength 5900 Å in the rest frame. He further assumed that the eye responds to wavelengths between 4000 and 8000 Å. Thus, his model exactly reverses the roles of star and eye as the "wide band" and "narrow band" component of the system. That in turn leads to the physically objectionable but artistically intriguing features of his predictions. Firstly, in his model any star which has an angle-dependent Doppler factor between 0.74 and 1.48 will be visible in the starbow but will be monochromatic with some pure color. In our model a blackbody star is still a blackbody at any Doppler factor, and can have an apparent color in the sequence of blackbody colors, but only if it is bright enough to stimulate cone vision. Secondly, in his model a star becomes invisible abruptly when its Doppler factor falls outside that narrow range. In our model the perceived brightness is a continuous function of Doppler factor, and in particular stars remain visible when blue-shifted by large amounts, up to $D = 5$ and beyond (see Fig. 4). Sänger's model for visibility is so extreme that it is actually independent of the transformation of intensity. A third feature implicit in Sänger's model was never mentioned by him. Aberration causes the starbow to contain an ever smaller portion of the starfield of the rest frame as the speed increases, so that the starbow becomes depopulated.

Other authors have suggested better approximations to predict visibility, but their work has not gained the wide circulation of Sänger's. Both Moskowitz[7] and Wertz[29] used a continuous spectrum for the star. Moskowitz used the experimental solar spectrum while Wertz considered the blackbody spectrum at various temperatures. They estimated the visibility from the one point on the emission spectrum which is Doppler shifted to the center of the visible range for the observer in the ship frame. Thus they agreed

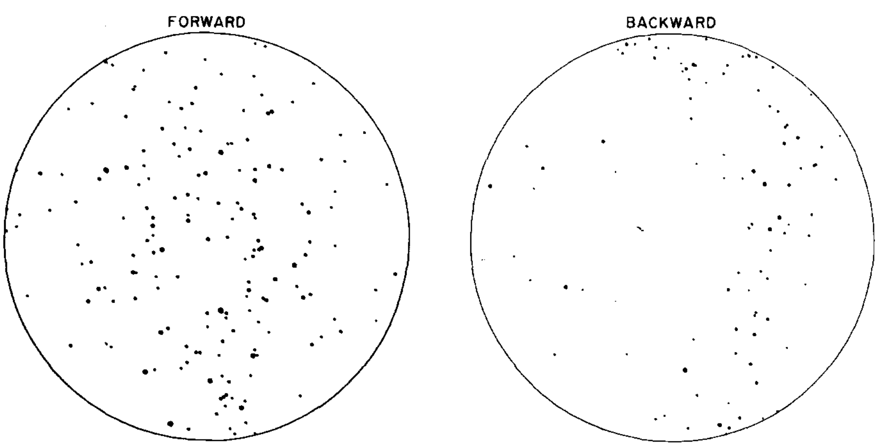

Fig. 6. Earth's starfield as viewed from a spaceship moving at speed $0.4c$ toward the north celestial pole.

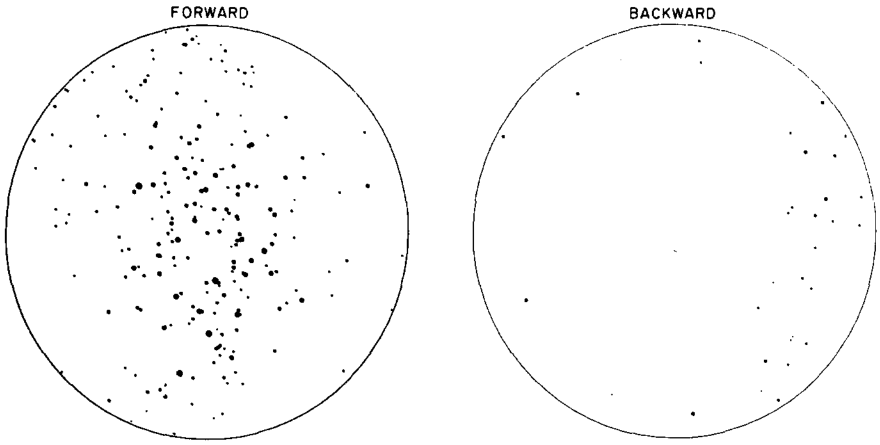

Fig. 7. Earth's starfield as viewed from a spaceship moving at speed 0.7c toward the north celestial pole.

that approaching the sun at speed 0.9c should cause it to appear dimmer by about 4 magnitudes. Wertz also predicted for example that a star at 32 000 K should appear brighter by about 4 magnitudes when the observer is receding at speed 0.9c.

By contrast, an inspection of our Fig. 4 or the use of Eq. (21) shows that approaching the sun (5800 K) at $\beta = 0.9$ ($D = 4.36$) causes it to brighten by 1.9 magnitudes. In fact all stars except the very hottest will appear brighter upon approach. The critical temperature for which brightness is maximum at rest is about 23 900 K. Thus a star at 32 000 K will appear to brighten slightly for recession up to about $\beta = 0.3$, but for recession at $\beta = 0.9$ it will appear dimmer by 1.1 magnitudes. This difference between our model and the earlier predictions arises mainly from the intensity transformation.

In Fig. 4 we also see that all stars become dimmer as the Doppler factor becomes very large. This suggests the possibility of "ultimate starbow", i.e., that Sänger's prediction of a ring of visible stars may come true but at higher speeds than those he considered. We prepared Fig. 9 to investigate this possibility. At this speed of 0.9998c the Doppler factor directly forward is 100, and the "transverse" Doppler factor (at $\theta = \pi/2$ in the rest frame) is 50. Thus for half of the starfield the very hot stars are 3.4–4 magnitudes dimmer than at rest, and even very cool stars are dimmer by 1-1.5 magnitudes from their maximum brightness at $D = 10$. We chose the magnification factor to be 6× so that the region of the Sänger starbow is included as the outer 30% of the radius of the projection.

We note in Fig. 9 that there are fewer star images still bright enough to be plotted. The mixture of brighter and dimmer star images, and the general pattern of their distribution, however, are not notably different from those at lower speeds. There is certainly not a ring of bright stars surrounding dimmer stars. There is only one star plotted within the limits of the Sänger starbow, a number which is consistent with the fact that at this speed the Sänger starbow represents only 0.74% of the sky of the rest frame.

Since the star table used here includes the brightest stars as seen from earth, it is biased toward the hot giant stars which are visible for great distances. It includes relatively few of the cool stars which actually dominate the stellar population. The cool stars are exactly those which would gain the greatest increase in apparent brightness due to the Doppler effect. If we had included a more representative population of stars, the forward cone at high speeds would be greatly enriched with bright star images from the added

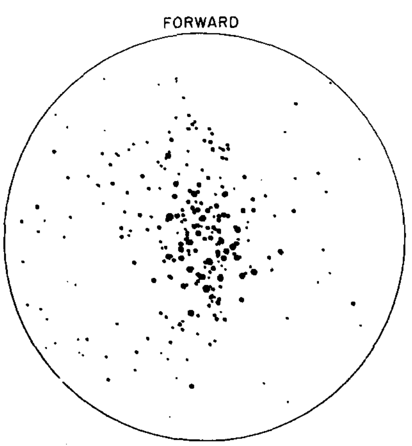

Fig. 8. Earth's starfield as viewed from a spaceship moving at speed 0.92c toward the north celestial pole.

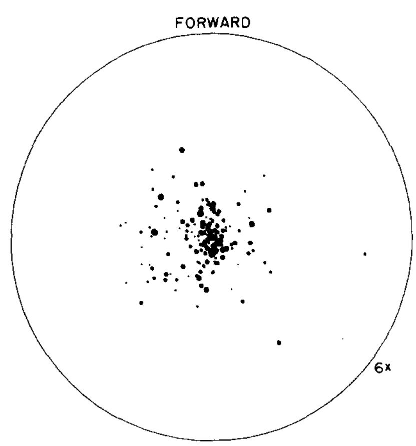

Fig. 9. Earth's starfield as viewed from a spaceship moving at speed 0.9998c toward the north celestial pole. Only the center of the forward hemisphere is shown, as it would appear through a 6× telescope. The disks to represent magnitudes are the same as used in Figs. 5–8.

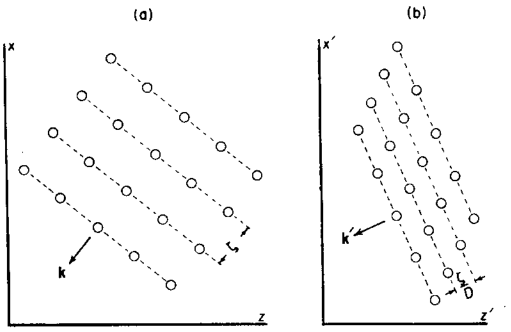

Fig. 10. (a) One plane of photons in frame S. The direction of propagation lies in this plane as indicated. (b) The same photons as measured in frame S'. Successive "fronts" of photons have their spacing changed by a factor $1/D$ as compared to frame S, but within each front the spacing is the same.

cool stars. Thus just as there is no starbow at speeds up to $0.9c$, there is no "ultimate starbow" at speeds closer to c.

We hope that this presentation will spell the end of the "starbow". At the same time we regret its demise. We have nothing so poetic to offer as its replacement, only better physics.

APPENDIX

Let us derive the transformation for the intensity of a beam of light using photons. For convenience we shall imagine that in frame S the beam consists of photons which at a single instant are located at the points of a rectangular lattice. We can take $\phi = 0$ so that the direction of propagation lies in the x-z plane, and the lattice is oriented on the direction of propagation. Figure 10(a) shows one plane containing photons. The photon spacing is taken to be ζ along the propagation direction, ξ across "fronts" in the plane shown, and η across "fronts" in the direction normal to the plane shown. The photon which is labeled by the integer indices $n_1n_2n_3$ follows the world line with equations:

$$x_{n_1n_2n_3} = n_1\xi\cos\theta + n_3\zeta\sin\theta - ct\sin\theta,$$

$$y_{n_1n_2n_3} = n_2\eta,$$

$$z_{n_1n_2n_3} = -n_1\xi\sin\theta + n_3\zeta\cos\theta - ct\cos\theta.$$

When we apply the Lorentz transformation [Eq. (1)] and aberration [Eqs. (5)–(8)], we find that the photon world line is described in frame S' by the equations

$$x'_{n_1n_2n_3} = n_1\xi\cos\theta' + n_3\zeta\gamma\sin\theta' - ct'\sin\theta',$$

$$y'_{n_1n_2n_3} = n_2\eta,$$

$$z'_{n_1n_2n_3} = -n_1\xi\sin\theta' + n_3\zeta\gamma(\cos\theta' - \beta) - ct'\cos\theta'.$$

By choosing $n_3 = 0$ in these expressions, we see that across a "front" the photons are spaced $\xi \times \eta$ in both frames. To find the spacing between "fronts" we take the scalar product of $\mathbf{r}'_{n_1n_2n_3}$ with the propagation vector $\mathbf{k}'$:

$$\begin{aligned}\mathbf{k}'\cdot\mathbf{r}'_{n_1n_2n_3} &= -k'\sin\theta' x'_{n_1n_2n_3} \\ &\quad + 0\cdot y'_{n_1n_2n_3} - k'\cos\theta' z'_{n_1n_2n_3} \\ &= k'[ct' - n_3\zeta\gamma(1-\beta\cos\theta')] \\ &= k'(ct' - n_3\zeta/D).\end{aligned}$$

Thus in frame S' the fronts are separated by ζ/D rather than by ζ as in frame S [Fig. 10(b)]. This is of course exactly analogous to the transformation of wavelength according to $\lambda' = \lambda/D$. This is also the equivalent of the transformation of the volume occupied by a classical wave packet, as employed by Einstein.[4] Altogether, the number of photons per unit volume in frame S' is D times the corresponding quantity in frame S. As they advance at speed c in either frame, there will be D times as many photons crossing unit area in unit time in frame S' as in frame S. Since each photon has D times as much energy in frame S' as in S, we have the overall intensity transformation $I' = D^2 I$. This result still applies when the photons are distributed randomly, because the transformation of photon spacings is linear.

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