Comment on "Further investigations of the operationally defined quantum phase"

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The normalization of the phase experiment of scheme 1 in the paper of Noh, Fougères, and Mandel [Phys. Rev. A **46**, 2840 (1992)] is inappropriate. The correct treatment of this measurement tends to the same result as in their scheme 2.

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Recently Noh, Fougères, and Mandel (NFM) suggested and performed in Ref. [1] an interesting quantum measurement. Even if the agreement between the measurement and its quantum-mechanical description is excellent, the interpretation of this result as a quantum phase measurement has some flaws. The problems associated with the discarding of noisy data, a well-behaved performance measure of phase detection, and with the discrete spectrum of the phase measurement, have to be taken into account [2]. These comments could also be applied to a continuation of the research of NFM [3], but this is not the aim of this Comment. Here we would like to point out the incorrect description of the measurement in scheme 1 performed on a beam splitter.

The problem is the following: If we wish to use the information obtained by two independent measurements, it is necessary to associate such measurements with commuting operators. The cosine and sine operators in scheme 1 of Ref. [3] are statistically independent, since the measurement of $\hat{C}$ does not affect the measurement of $\hat{S}$ and vice versa. Really, we can measure at first the photon numbers $\hat{n}_3, \hat{n}_4$ and derive the statistics of $\hat{S} \equiv \hat{n}_3 - \hat{n}_4$, and afterwards, we can repeat the measurement with an inserted $\lambda/4$ shifter, yielding the statistics of $\hat{n}_5, \hat{n}_6$ and $\hat{C} \equiv \hat{n}_5 - \hat{n}_6$. There are no doubts that the resulting probabilities will be the same, if we invert the succession and measure $\hat{C}$ with an inserted $\lambda/4$ shifter before measurement of $\hat{S}$. Measurements of $\hat{S}$ and $\hat{C}$ should be represented therefore as commuting operators and Eqs. (8) and (9) of Ref. [3] should be replaced by

$$\hat{S} = -i(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2) \tag{1}$$

and

$$\hat{C} = \hat{b}_2^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_2, \tag{2}$$

where $\hat{a}_{1,2}, \hat{b}_{1,2}$ are independent boson operators, i.e.,

$$[\hat{a}_i, \hat{a}_j] = [\hat{b}_i, \hat{b}_j] = [\hat{a}_i, \hat{b}_j] = [\hat{a}_i, \hat{b}_j^\dagger] = 0$$

and

$$[\hat{a}_i, \hat{a}_j^\dagger] = [\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}.$$

To keep the interpretation of the measurement as quantum phase detection, the field in the b modes has to be assumed to be identical with that in the a modes, i.e.,

$$|\psi\rangle_{\text{in}} = |u_1\rangle_{a1} \otimes |u_2\rangle_{a2} \otimes |u_1\rangle_{b1} \otimes |u_2\rangle_{b2}, \tag{3}$$

u_1, u_2 being the complex amplitudes of the coherent input fields.

The measured complex amplitude is given in analogy with the previous treatment [2] as

$$\hat{Y} = \hat{C} + i\hat{S}, \tag{4}$$

and the exponential phase operator relevant to this measurement is therefore

$$\hat{R} = \begin{cases} \hat{Y}\,(\hat{Y}^\dagger\hat{Y})^{-1/2}, & \text{for } \mathcal{H}_0^\perp, \\ 0, & \text{for all states in } \mathcal{H}_0. \end{cases} \tag{5}$$

$\mathcal{H}_0$ means the subspace spanned by the eigenstates with the zero complex amplitude $\{|y = 0\rangle\}$ and $\mathcal{H}_0^\perp$ is its orthogonal complement. The moments of this operator are important for evaluation of the accuracy of the phase measurement, since the performance measure used in the paper [3] $\langle(\Delta C_M)^2\rangle + \langle(\Delta S_M)^2\rangle$ is equal to the dispersion

$$D^2 = 1 - |\langle\hat{R}\rangle|^2. \tag{6}$$

The normalized moments of the cosine and sine operators may be simply measured and theoretically evaluated as

$$\begin{aligned} \langle\hat{S}_M^r\rangle &= \left\langle \frac{(m_3 - m_4)^r}{[(m_4 - m_3)^2 + (m_6 - m_5)^2]^{r/2}} \right\rangle, \\ \langle\hat{C}_M^r\rangle &= \left\langle \frac{(m_5 - m_6)^r}{[(m_4 - m_3)^2 + (m_6 - m_5)^2]^{r/2}} \right\rangle, \end{aligned} \tag{7}$$

where m_k are the measured outputs on the photodetectors, and power $r = 1$ is used for the case of dispersion (6). Let us emphasize that the measurements yielding $m_3 = m_4$ and simultaneously $m_5 = m_6$ do not contribute to the average values (6) and (7) and should be omitted without any renormalization according to the expression (5). We also could discard these noisy data, as is done in Ref. [1], but then the phase measurement should be interpreted as a phase measurement in a state different from the input state [2].

The discrepancy between our proposal and the treatment of Ref. [3] is in the following:

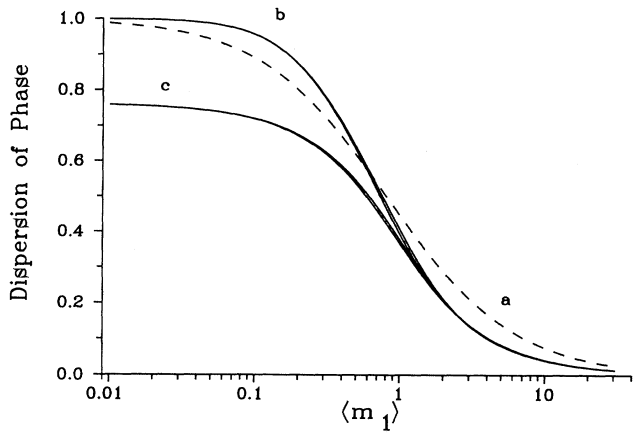

FIG. 1. Dispersions D as a function of the mean number of detected photons $\langle m_1 \rangle$ for coherent fields with $\langle m_2 \rangle / \langle m_1 \rangle = 1.6$. The parameters are the same as in Fig. 6 of Ref. [3]. The dashed curve a represents the NFM result. The curve b is the theoretical prediction of this Comment computed on the basis of Eq. (7). The curve c is the same with discarding of noisy data according to Refs. [1, 2].

(1) The measurement in scheme 1 was described incorrectly, since the operators associated with the independent measurements do not commute.

(2) The measurement in scheme 1 is fully equivalent to the measurement in scheme 2 and both have to be described using four modes. Let us introduce the two-mode coherent states on the output as

$$|\varphi_1(u_1,u_2)\rangle = \left| \frac{u_1 + iu_2}{\sqrt{2}} \right\rangle_3 \otimes \left| \frac{u_1 - iu_2}{\sqrt{2}} \right\rangle_4$$

and

$$|\varphi_2(u_1,u_2)\rangle = \left| \frac{u_1 + u_2}{\sqrt{2}} \right\rangle_5 \otimes \left| \frac{u_1 - u_2}{\sqrt{2}} \right\rangle_6 .$$

Then the photon counting measurement performed on the states: (a) $|\varphi_1(u_1,u_2)/\sqrt{2}\rangle$ and $|\varphi_2(u_1,u_2)/\sqrt{2}\rangle$ yields information about *sine* and *cosine* in scheme 1; (b) $|\varphi_1(u_1,iu_2)/\sqrt{2}\rangle$ and $|\varphi_2(u_1,iu_2)/\sqrt{2}\rangle$ yields information about *cosine* and *sine* in scheme 2 [4], which effectively tends to the same result. The only problem of scheme 1 is whether it is experimentally possible to reproduce the same quantum states during the independent measurements of sine and cosine of phase with sufficient accuracy. The identity of states is important for the interpretation of the performed quantum detection as a quantum phase measurement. This problem disappears in scheme 2, since these measurements are performed in the same time.

(3) The inappropriate treatment of normalization in Eq. (59) of Ref. [3] tends to a value two times higher than our result in the classical limit for $u_{1,2} \gg 1$. Particularly, we could easily conclude that the highest term of dispersion is

$$D^2 \approx \frac{1}{4}\left[\frac{1}{|v_1|^2} + \frac{1}{|v_2|^2}\right] + O(1/|v|^4).$$

This conclusion follows also from the previous item (2) of this Comment and together with the limiting treatment of Ref. [1]. Let us emphasize that the measurement after scheme 1 needs an energy two times higher than after scheme 2. The expected difference between the result of Ref. [3] and our suggestion is given in Fig. 1. Curve a represents the NFM result and the curve b our correction. The dependence on the measured phase difference is very weak (less than 3%), as indicated by the splitting of the curve. Curve c represents our prediction for measurement with the discarding of ambiguous data [1]. Of course, the measurement should be interpreted as a quantum phase measurement in a state different from the coherent one [2].

(4) Since the normalization is matched to the first moments only, the problems appear here for higher moments, which can exceed unity; see Sec. VIII of Ref. [3]. This is not possible in the case of the regular quantum-mechanical treatment represented by the expression (7). The excellent experimental and theoretical agreement reported in the Fig. 6 of Ref. [3] is therefore not sufficient for the correct interpretation, but indicates that regular treatment also will tend to the coincidence between the theory and measurement.

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