Estimation of counted quantum phase

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The estimation of phase shift in realistic experiments is addressed here. Provided that the phase concept is not ideal, the phase shift should always be inferred from the performed measurement. Particularly, two different estimations are suggested. The conditional probability is given by the experimentally observable dependence of the measured probability on the phase shift. As an explicit example, the accuracy of the phase measurement on the Mach-Zehnder interferometer is evaluated using different methods for comparison.

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The problem of quantum phase is as old as quantum mechanics itself [1], but new techniques of phase measurement appeared recently in connection with the rapid progress in quantum optics. Achievements in technology of generation and detection of nonclassical light triggered various schemes of feasible phase detection [2–18], to cite, without requirements for completness, at least some titles from the existing bibliography. Significantly, recent theoretical and experimental investigations of Noh, Fougères, and Mandel [11–13] started the wave of renewed interest in this topic. Their operational approach is based on multimode homodyne detection, where the phase shift is derived from photon counting using four detectors. More generally, any quantum phase operator may be associated with the simultaneous measurement of two commuting Hermitian operators [19]. The ideal quantum phase concepts can be distinguished as the reducible representations of a Euclidean algebra on Hermitian operators. Nevertheless, such measurement requires a continuous spectrum of registered observables and can be achieved only as the limiting case. In the realistic quantum phase measurement information about the phase shift is obtained via counting of photons and the measured phase parameter is therefore in principle *discrete*. On the other hand, the induced phase shift is a *continuous* parameter. The measured output therefore cannot be straightforwardly interpreted as possible values of phase shift. This is the typical situation in quantum optics with realistic finite energy fields [11–13,16,20,21].

In this paper, the realistic quantum measurement will be completed by the quantum estimation of phase shift. Particularly, two conditional probability distributions of inferred phase shift will be suggested as an alternative to the shift-invariant estimation proposed recently [22]. General theory will be illustrated on the example of the Mach-Zehnder interferometer, where the differences between proposed and shift-invariant estimations are apparent. The quantum estimation procedure is crucial for phase concepts based on the detection of a different-than-phase variable [17,18,23], where the phase shift is inferred rather than measured. On the other hand, only the ideal quantum phase concepts yield information about phase shift directly without any quantum estimation.

Let us describe the operator (Heisenberg) picture of the discrete detection of a phaselike variable using advantageously the formalism of generalized measurement [24]. Since the treatment is formally analogous to the continuous case [19], let us point out the main differences only. Suppose that the general measurement of a pair of observables represented by commuting Hermitian operators $\hat{Y}_1, \hat{Y}_2$, $[\hat{Y}_1, \hat{Y}_2] = 0$ is available. Assuming the discrete spectrum of the complex-valued operator $\hat{Y} \equiv \hat{Y}_1 + i\hat{Y}_2$, the relations of completeness and orthogonality read $\sum_k |y_k\rangle\langle y_k| = \hat{1}$, where $|y_k\rangle$ are the orthonormal and complete eigenvectors $\hat{Y}|y_k\rangle = y_k|y_k\rangle$, $\langle y_k|y_l\rangle = \delta_{kl}$. Here, for the sake of brevity, the notation does not show explicitly the possible degenerations of the eigenvectors $|y_k\rangle$. The probability distribution of finding the complex amplitude y_k by performing the measurement on a general quantum state $|\psi\rangle$ is given by $p(y_k) = |\langle\psi|y_k\rangle|^2$. The purpose of an arbitrary phase detection is to determine the nonrandom c-number displacement parameter $\eta \in (-\pi, \pi]$ entering the displacement transformation [24] of quantum state as $|\psi(\eta)\rangle = e^{-i\eta\hat{N}}|\psi\rangle$, $\hat{N}$ being a Hermitian displacement operator. The variable η represents the true value of the phase shift and the quantum phase theory should determine this *a priori* unknown parameter as accurately as possible. The quantum measurement specified above directly yields statistical information about the discrete phaselike variable $\hat{\varphi} = \arg[\hat{Y}]$. The dispersion of a detected phaselike variable [25] is given as $D^2 = 1 - |\langle e^{i\varphi}\rangle|^2$, where the exponential operator of the measured phaselike variable is

$$\hat{R} = \begin{cases} \hat{Y}\,(\hat{Y}^\dagger\hat{Y})^{-1/2} & \text{for } \mathcal{H}_0^\perp \\ 0 & \text{for all states in } \mathcal{H}_0. \end{cases} \tag{1}$$

Here, $\mathcal{H}_0$ is the subspace spanned by the eigenstates with zero complex amplitude $\hat{Y}|y=0\rangle = 0$, $\mathcal{H}_0^\perp$ being the orthogonal complement of $\mathcal{H}_0$. Handling data with indefinite phase could be justified as in Ref. [19]. Nevertheless, at this point, the formal analogy between discrete and continuous models comes to an end. We should infer the statistics of continuous phase shift variable ϕ from the

*On leave of absence from Laboratory of Quantum Optics, Palacký University Svobody 26, 771 46 Olomouc, Czech Republic. Electronic address: hradil@risc.upol.cs knowledge of measured (discrete) phaselike output. The measured phaselike variable $\varphi_k \in (-\pi, \pi]$ can achieve only the discrete values φ_k consistent with the exponential operator (1). The probability for this to happen can be easily expressed as the sum of all the contributions p_m, for which $\arg y_m = \varphi_k$. The conditional probability distribution of measuring the value φ_k when η is true is then given as

$$p_k(\eta) = \sum_{\arg y_m = \varphi_k} |\langle y_m | e^{-i\eta \hat{N}} \psi \rangle|^2. \qquad (2)$$

This distribution is normalized with respect to the parameter k for each phase shift η, $\sum_k p_k(\eta) = 1$. The Bayes rule [25] may now be applied, asking what actual value of phase shift ϕ could be inferred if the phase φ_k was detected,

$$p(\phi|\varphi_k) = \frac{1}{C_k}\, p_k(\phi), \qquad (3)$$

normalization being $C_k = \int_{-\pi}^{\pi} d\theta\, p_k(\theta)$. This distribution reflects the conditional probability of phase shift after a single measurement without any prior knowledge. The phase information may be associated with the conditional probability of inferring the phase shift ϕ when the true value is η,

$$P_1(\phi|\eta) = \sum_k \frac{1}{C_k}\, p_k(\phi)\, p_k(\eta), \qquad (4)$$

representing the first proposal of this paper. Let us emphasize that phase ϕ is not a quantum mechanical variable, but the estimation obtained from the measurements. The quantum state of the field influences the form of the distributions $p_k(\eta)$, but not the prior knowledge. The known dependence of the measured statistics on the phase shift $p_k(\eta)$ fully determines the statistics of the inferred phase shift $P_1(\phi|\eta)$. The inferred probability density is symmetric in the true and inferred phase shifts, but is not shift invariant, it is dependent only on the difference $\eta - \phi$. Shift-invariant estimation was suggested by Noh, Fougères, and Mandel [22]. They inferred the phase shift as ϕ whenever for the true phase shift $\eta + \theta$ the output $\varphi_k = \phi + \theta$ is registered, θ being an arbitrary phase shift. Consequently, the resulting probability distribution representing the continuous limit of the treatment of Noh, Fougères, and Mandel is given as

$$P_{NFM}(\phi|\eta) = \frac{1}{2\pi} \sum_k p_k(\eta - \phi + \varphi_k). \qquad (5)$$

Both predictions (4) and (5) tend to comparable results, if the quantum measurement (2) itself behaves in a shift-invariant manner. Nevertheless, significant differences can be expected, if the dispersion of a detected phaselike variable strongly depends on the true value of phase shift η. We will demonstrate this feature using the example of the Mach-Zehnder interferometer. But before doing this let us formulate a more effective estimation, which updates the prior knowledge inferring the phase shift after each trial in dependence on the measured output.

Let us suppose that the measurement of φ_{k_i} is performed repeatedly n times; $i = 1, 2, ..., n$, under identical conditions (in the given quantum state for the same true value of phase shift η). The measured output φ_{k_i} should be interpreted as the estimation of the phase shift with the probability distribution $p(\phi|\varphi_{k_i})$ in accordance with relation (3). This information may be used as the prior probability distribution for the subsequent $(i+1)$th measurement. Then for the uniform prior distribution before the first trial, the probability distribution after the nth measurement is given as the normalized product (likelihood function)

$$P_2(\phi|\eta) \propto \prod_{i=1}^{n} p(\phi|\varphi_{k_i}) \qquad (6)$$

and the actual phase shift may be determined using maximum likelihood estimation [26–28]. The dependence on the true value of phase shift η is hidden in the statistics of measured outputs φ_{k_i}, and may be shown explicitly for sufficiently large number of trials n. Then each value φ_{k_i} appears about $np_{k_i}(\eta)$ times in the likelihood function and the distribution (6) reads

$$P_2(\phi|\eta) = \frac{1}{C_n(\eta)} \left\{ \prod_k [p_k(\phi)]^{p_k(\eta)} \right\}^n, \qquad (7)$$

where the normalization is

$$C_n(\eta) = \int_{-\pi}^{\pi} d\phi \left\{ \prod_k [p_k(\phi)]^{p_k(\eta)} \right\}^n.$$

Index k exhausts all the possible values φ_k appearing with nonzero probability. This is the second proposal for how to infer the statistics of phase shift from the performed measurement. Since the phase information is accumulated in the process of measurement, this estimation is more efficient than estimation based on a single measurement. The needed total expenses are n times larger than in the case of single trial. The resulting conditional probability distribution $P_2(\varphi|\eta)$ is neither shift-invariant nor symmetric. However, for sufficiently large number of repetition n (the so called "Braunstein's knee"), the distribution is becoming Gaussian with the resolution corresponding to the Fisher information [29].

Let us apply our theory to the description of the SU(2) interferometer [8]. Similar quantum detection of phase shift was already addressed in Ref. [16]. The ordinary measurement setup is depicted in Fig. 1. Let us restrict ourselves to the case of single-mode quasimonochromatic fields on each port for simplicity. Within the domain of classical optics, the input fields with complex amplitudes V_1, V_2 are combined on the 50-50 ideal beam splitter and the phase shift η is induced in the arms of the interferometer. Assuming the closed auxiliary input port 2 $V_2 = 0$, the integrated intensities on output ports W_3, W_4 may be easily specified as $W_3 = W_1 \cos^2 \eta/2$, $W_4 = W_1 \sin^2 \eta/2$, where $W_1 = \xi \int_t^{t+T} dt'\, |V_1(t')|^2$ and ξ is the efficiency of the detector. In the classical interferometry, the average value of energy is registered on each of output ports 3

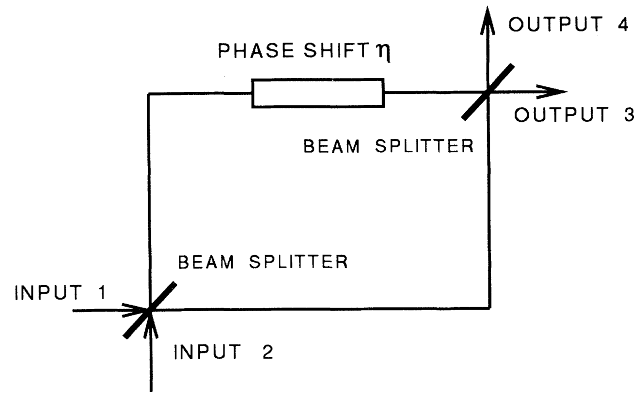

FIG. 1. Setup of Mach-Zehnder interferometer.

and 4 in dependence on the phase shift. The phase difference η is then inferred from this interferometric pattern. The accuracy is given in semiclassical approximation by the fluctuations of photon number [4,5,8]. The quantum counterpart may be easily established assuming the classical relation

$$\tan\eta/2 = \sqrt{\frac{W_4}{W_3}}. \tag{8}$$

The desired phase shift can be deduced from the phase of the complex amplitude $Y_{MZ} = W_3 + iW_4$. Assuming canonical quantization and an ideal process of photodetection, a quantum model may be obtained by formally replacing complex amplitudes and integrated intensities by annihilation operators $\hat{a}_i$ and photon number operators $\hat{N}_i$, respectively, $i = 1,2,3,4$. The quantum analogy of the closed auxiliary port 2 is then the condition $\langle\hat{a}_2\rangle = 0$. The complex amplitude Y_{MZ} may be quantized introducing advantageously the generators of the SU(2) algebra [8]. Let us define $\hat{J}_1 = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger)$, $\hat{J}_2 = \frac{1}{2i}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1\hat{a}_2^\dagger)$, $\hat{J}_3 = \frac{1}{2}(\hat{n}_1 - \hat{n}_2)$, and the Casimir operator as $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 = \hat{C}(\hat{C}+1)$, where $\hat{C} = \frac{1}{2}(\hat{n}_1 + \hat{n}_2)$. Then the measured complex amplitude is

$$\hat{Y}_{MZ} = \hat{N}_3 + i\hat{N}_4 = (1+i)\hat{C} + (1-i)\exp(-i\eta\hat{J}_2) \times\hat{J}_3\exp(i\eta\hat{J}_2), \tag{9}$$

the displacement operator $\hat{N}$ being $\hat{J}_2$. The measured phaselike variable φ reproducing in the classical limit the relation (8) should be defined as

$$\tan^2\frac{\varphi}{2} = \tan(\arg Y_{MZ}). \tag{10}$$

The sine and cosine are given using the measured photoelectron counts N_3, N_4 as

$$\sin\varphi = \pm\frac{2\sqrt{N_3N_4}}{N_3+N_4}, \qquad \cos\varphi = \frac{N_3-N_4}{N_3+N_4}. \tag{11}$$

Obviously, this detection cannot distinguish the sign of φ. Let us specify for concreteness the quantum measurement in the state with closed auxiliary input port 2 and N photons on the input port 1, $|\psi\rangle_{in} = |N\rangle_1 \otimes |0\rangle_2$. The joint probability distribution of counting $N_3 = N-k$ and $N_4 = k$ is given by binomial distribution,

$$\mathcal{P}(N_3 = N-k, N_4 = k) = \binom{N}{k}\left[\cos^2\frac{\eta}{2}\right]^{N-k}\left[\sin^2\frac{\eta}{2}\right]^k. \tag{12}$$

Detection of $k = 1,2,...,N$ photons on port 4 should be interpreted as registration of phases φ_k and $\varphi_{-k} = -\varphi_k$,

$$\varphi_k = 2\arctan\sqrt{\frac{k}{N-k}}. \tag{13}$$

Phases with different sign are detected with equal probabilities $p_k(\eta) = p_{-k}(\eta) = \frac{1}{2}\mathcal{P}(N-k,k)$. Zero phase variable $\varphi_0 = 0$ for $k = 0$ is detected with the conditional probability $p_k(\eta) = \mathcal{P}(N,0)$. Consequently, the sharpest distribution appears when the true phase shift is $\eta = 0$, since only the value $\varphi_k = 0$ is registered, $p_k(\eta = 0) = \delta_{k0}$. On the contrary, the broadest distribution appears at $\eta = \pi/2$.

Let us specify explicitly the case of sharpest phase measurement. The inferred conditional probability distribution (4) reduces to the only nonzero term, yielding

$$P(\phi|\eta = 0) = \frac{(2N)!!}{2\pi\ (2N-1)!!}[\cos\phi/2]^{2N}. \tag{14}$$

Dispersion of inferred phase shift is simply

$$D_{MZ}^2 = 1 - \left|\int_{-\pi}^{\pi} d\phi\, e^{i\phi}P(\phi|\eta = 0)\right|^2 = \frac{2N+1}{(N+1)^2} \approx \frac{2}{N}, \tag{15}$$

in accordance with the accuracy of the classical interferometry. One can also easily verify that the estimation (7) tends to the same probability distribution in this special case. Of course, the number of particles in each trial should be renormalized as N/n to compare the performance with the same total energy N.

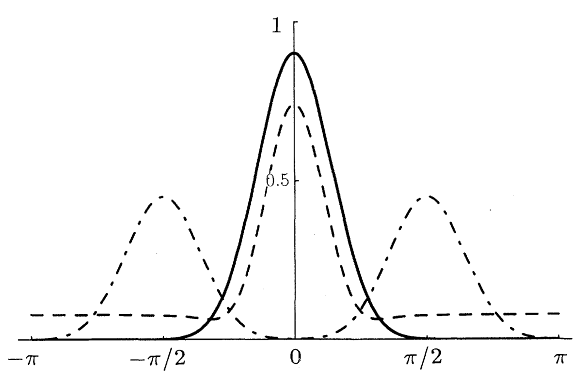

FIG. 2. The inferred probability distributions $P_1(\phi|\eta = 0)$ (full line), $P_1(\phi|\eta = \pi/2)$ (dot-dashed line), and $P_{NFM}(\phi)$ for $\eta = 0$ (dashed line) for $N = 10$.

The estimation (5) exhibits different behavior, yielding the equal shape of probability distribution for any true value of phase shift, independent of the actually detected statistics $p_k(\eta)$. The explicit expression is

$$P(\alpha) = \frac{1}{4\pi} \sum_{k=0}^{N} \binom{N}{k} \left\{ \left[\cos^2 \frac{\alpha - \varphi_k}{2}\right]^{N-k} \left[\sin^2 \frac{\alpha - \varphi_k}{2}\right]^{k} + \left[\cos^2 \frac{\alpha + \varphi_k}{2}\right]^{N-k} \left[\sin^2 \frac{\alpha + \varphi_k}{2}\right]^{k} \right\}, \qquad (16)$$

where φ_k is defined in (13) and $\alpha = \phi - \eta$. Obviously, the distribution exhibits one peak, even if the measured values and inferred distribution (4) show two peaks. Numerical comparison of both the estimations for the case of sharpest measurement $\eta = 0$ and for $\eta = \pi/2$ is given in Fig. 2. Of course, this rough discrepancy between both the estimations disappears provided that phase shift is estimated on the interval $(0, \pi)$ only.

We have explicitly demonstrated that the discrete variable cannot be directly used as a good estimator of continuous phase shift. Detection of any phase-sensitive (continuous or discrete) variable may be in principle used for evaluation of phase shift. The argumentation and the derived probability distributions (4) and (7) do not change, if we disregard the phaselike character of measured output assuming that k represents a general phase-sensitive variable. Phase shift is always partially measured and partially inferred [16–18], even in the case of classical interferometry. Phase measurement represents one of the most accurate detection techniques currently available, and the investigation of noises involved therefore deserves corresponding attention.

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