Decomposition of the electromagnetic field in lossless inhomogeneous dispersive dielectrics

Z. Hradil
Department of Optics, Palacký University, 17. listopadu 50, 772 07 Olomouc, Czech Republic
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The interaction of the electromagnetic field with lossless Lorentz oscillators modeling the inhomogeneous dielectrics is diagonalized in a closed Fabry-Pérot resonator using the generalized polariton transformation. Resulting dispersion relations coincide with the classical ones obtained by the solution of the wave equation. The corresponding decomposition is, however, orthogonal and complete in the enlarged Hilbert space.

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I. INTRODUCTION

The investigation of the electromagnetic field in dielectrics has attracted growing attention recently [1–5]. Particularly, quantum aspects of this problem are of current interest due to the potential applications in technology of nanostructures. This research includes investigation of quantum wells embedded in microcavities [6–8] and generation and propagation of nonclassical states of light. Hence the quantization of the electromagnetic field in inhomogeneous dispersive linear dielectrics represents a nontrivial problem [9–11]. In this paper the canonical quantization scheme formulated by Huttner and Barnett [5] for homogeneous dielectrics will be extended to the case of lossless dispersive inhomogeneity. The method will be illustrated in the one–dimensional problem of quantization of the electromagnetic field in a closed cavity with dispersive inhomogeneity modeled by the finite superposition of lossless Lorentz oscillators. Normal modes are associated with (generalized) polariton transformation. Polariton solution exactly yields the orthogonal decomposition.

The considerations presented by this paper are motivated by standard electromagnetic theory [12]. Suppose for concreteness the geometry of a closed Fabry–Pérot resonator with dispersive inhomogeneity [refractive index $n(\Omega)$] along the z axis, as sketched in Fig. 1. For the sake of simplicity the s polarization of the electric field will only be assumed in the following. Using the Maxwell equations, the eigenmodes of the cavity with the time dependence $e^{i\Omega t}$ may be specified as a solution of the Helmholtz (time-independent) wave equation

$$\left[\Delta+\frac{\Omega^2}{c^2}[1+\theta(z)\chi(\Omega)]\right]E=0, \tag{1}$$

where $\chi(\omega)$ is the susceptibility. The inhomogeneity is included in the characteristic function

$$\theta(z)=\begin{cases}1 & \text{for } |z|\leq l/2\\ 0 & \text{for } |z|>l/2.\end{cases}$$

Imposing the boundary conditions at $\pm L/2$ as a perfect reflection at the mirrors $E(\pm L/2)=0$, the solutions of Eq. (1) represent an eigenmode decomposition in classical electromagnetic theory. In general, susceptibility should be treated as a complex-valued function of frequency due to the Kramers–Kronig relations [13]

$$\chi'(\Omega)=\frac{2}{\pi}\int_0^\infty d\omega\,\frac{\omega\chi''(\omega)}{\omega^2-\Omega^2},$$

$$\chi''(\Omega)=-\frac{2\Omega}{\pi}\int_0^\infty d\omega\,\frac{\chi'(\omega)}{\omega^2-\Omega^2},$$

$\chi=\chi'+i\chi''$; relating the real and imaginary parts by Hilbert transformation. In such a case the eigenmode decomposition (1) does not exist either for homogeneous cavity, since any propagating wave is damped. Nevertheless eigenmodes may be found providing that the imaginary part of susceptibility disappears. Hence the susceptibility may be considered as real and Eq. (1) may be simply solved in the regions where the coefficients are continuous functions of z. The electric field E must be continuous together with its first derivation $\partial E/\partial z$ for each $|z|\leq L/2$, yielding the dispersion relation

$$Q_z'\tan\left[Q_z\frac{L-l}{2}\right]=Q_z\cot\left[Q_z'\frac{l}{2}\right], \tag{2}$$

$$Q_z'\tan\left[Q_z\frac{L-l}{2}\right]=-Q_z\tan\left[Q_z'\frac{l}{2}\right],$$

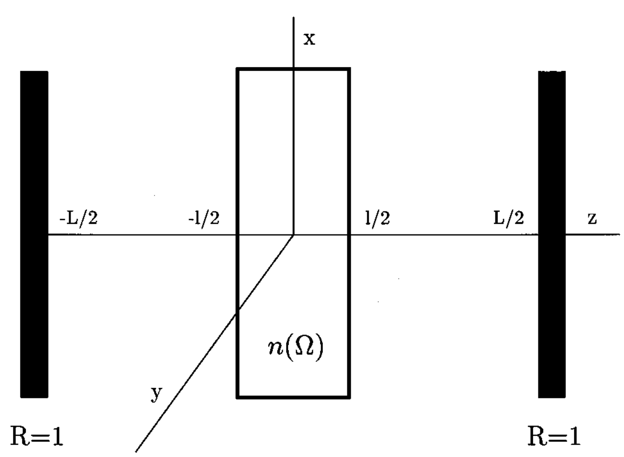

FIG. 1. Geometry of a closed cavity with dispersive inhomogeneity in the z direction.

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where $Q_z^2=(\Omega^2/c^2)-q^2$ and $Q'^2_z=Q_z^2+\chi(\Omega)\Omega^2/c^2$. Here $\mathbf{q}$ represents the two-dimensinal (2D) component of a wave vector parallel to the boundaries. This transcendent equation may be solved yielding the discrete set of eigenvalues. Nevertheless, a simple analysis shows that the corresponding eigenfunctions $E_{q,m}(z)$ are not orthogonal since the ''potential'' $\chi(\Omega)$ depends on the frequency. This is the source of theoretical troubles, since the decomposition of the electric field is becoming questionable. The quasinormal modes in a leaky macrocavity were used in Refs. [14]. Quantization of this scheme will be given in the following section.

II. CANONICAL QUANTIZATION

Let us formulate a canonical description of the interaction of a transversal electromagnetic field with matter. Neglecting other losses the the Lagrangian reads $L=\int d^3\mathbf{r}\mathscr{L}(\mathbf{r})$, where the Lagrangian density is $\mathscr{L}=\mathscr{L}_{em}+\mathscr{L}_{mat}+\mathscr{L}_{int}$,

$$\mathscr{L}_{em}=\frac{\epsilon_0}{2}[\dot{\mathbf{A}}^2-c^2(\boldsymbol{\nabla}\times\mathbf{A})^2], \tag{3}$$

$$\mathscr{L}_{mat}=\frac{\rho}{2}[\dot{\mathbf{X}}^2-\omega_0^2\mathbf{X}^2], \tag{4}$$

$$\mathscr{L}_{int}=-\alpha\mathbf{A}\cdot\dot{\mathbf{X}}. \tag{5}$$

Boldface characters denote vectors and the overdot means time derivation $\partial/\partial t$. The electromagnetic part is represented by vector field $\mathbf{A}$ defined in the whole cavity. The polarization part is modeled by a harmonic-oscillator field (Lorentz model) with amplitude vector $\mathbf{X}$, which is nonzero only in the interval of the inhomogeneity $|z|\leqslant l/2$, ω_0 being the frequency and ρ the density. Interaction of both fields is characterized by interaction constant α. For simplicity, linearly polarized fields with polarization parallel to discontinuity planes $z=\pm l/2$ will be assumed (s polarization) in the following. The vector field $\mathbf{A}$ may be interpreted as electric intensity and both fields may be represented as (real) scalar fields. The Lagrange-Euler equations then yield the wave equation (1) with the susceptibility

$$\chi'=\frac{\alpha^2}{\epsilon_0\rho(\omega_0^2-\Omega^2)},\quad \chi''=-\frac{\pi\alpha^2}{\epsilon_0\rho}\delta(\omega_0^2-\Omega^2). \tag{6}$$

The interacting fields may be quantized using expansion in orthogonal basis relevant to respective free fields. A standard approach [5] may be used in the plane xy perpendicular to the direction of inhomogeneity, since $\mathbf{q}$ is conserved due to the translation symmetry. In the z direction, the eigenfunction for the light and matter excitation parts should be distinguished. Assuming $e^{i\mathbf{qs}}/(2\pi)$ dependence, $\mathbf{s}$ being the projection of the 3D $\mathbf{r}$ vector into the xy plane, $\{\varphi_m(z)\}$ are the solutions of the time-independent wave equation

$$\left[\frac{d^2}{dz^2}+Q_m^2\right]\varphi_m(z)=0;\quad Q_m^2=\frac{\Omega_m^2}{c^2}-q^2, \tag{7}$$

fulfilling the given boundary conditions at $|z|\leqslant L/2$. Assuming for concreteness perfect reflection on the end mirrors, we have $Q_m=\pi m/L$; $m=1,2,3,\ldots$. Frequencies are quantized as $\Omega_m(q)=c\sqrt{(\pi m/L)^2+q^2}$, $q=|\mathbf{q}|$. Corresponding eigenfunctions are given as $\varphi_m(z)=\sqrt{2}\sin[m\pi(z/L+1/2)]$. The inhomogeneity of matter excitations is included in the definition of eigenfunctions $\chi_{\mathbf{q},\xi}(\mathbf{r})$, since they are nonzero only in the interval $|z|\leqslant l/2$. Two sets of functions $\varphi_{\mathbf{q},m}(\mathbf{r})$ and $\chi_{\mathbf{q},\xi}(\mathbf{r})$ defined in the 3D space are orthogonal and complete in the volumes of quantization

$$\int d^3\mathbf{r}\varphi_{\mathbf{q},m}(\mathbf{r})\varphi^*_{\mathbf{q}',n}(\mathbf{r})=\delta(\mathbf{q}-\mathbf{q}')\delta_{m,n},$$

$$\sum_m \int d^2\mathbf{q}\varphi_{\mathbf{q},m}(\mathbf{r})\varphi^*_{\mathbf{q},m}(\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int d^3\mathbf{r}\chi_{\mathbf{q},\xi}(\mathbf{r})\chi^*_{\mathbf{q}',\eta}(\mathbf{r})=\delta(\mathbf{q}-\mathbf{q}')\delta_{\xi,\eta},$$

$$\sum_\xi \int d^2\mathbf{q}\chi_{\mathbf{q},\xi}(\mathbf{r})\chi^*_{\mathbf{q},\xi}(\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')\theta(z).$$

The cross products are given by the matrix elements

$$\int \varphi_{\mathbf{q},m}(\mathbf{r})\chi^*_{\mathbf{q}',\xi}(\mathbf{r})d^3\mathbf{r}=\delta(\mathbf{q}-\mathbf{q}')K_{m,\xi},$$

$$K_{m,\xi}=\int_{-l/2}^{l/2}\varphi_m(z)\chi_\xi(z)dz,\quad K_{m,\xi}=K^*_{m,\xi}.$$

The functions $\{\chi_\xi(z)\}$ are orthogonal and complete in the interval $|z|\leqslant l/2$. The explicit form of these functions is not important in our considerations. In the following the decomposition of the electromagnetic field will be consistently enumerated by Latin indices, whereas Greek ones will be used for the matter oscillations. The classical fields may be decomposed as

$$A(\mathbf{r},t)=\frac{1}{2\pi}\sum_m \int d^2\mathbf{q}A_{\mathbf{q},m}(t)\varphi_{\mathbf{q},m}(\mathbf{r}),$$

$$X(\mathbf{r},t)=\frac{1}{2\pi}\sum_\xi \int d^2\mathbf{q}X_{\mathbf{q},\xi}(t)\chi_{\mathbf{q},\xi}(\mathbf{r}).$$

The total Lagrangian $L=L_{em}+L_{mat}+L_{int}$ may be quantized as

$$L_{em}=\epsilon_0\sum_m \int' d^2\mathbf{q}[|\dot{A}_{\mathbf{q},m}|^2-\Omega_m^2(q)|A_{\mathbf{q},m}|^2], \tag{8}$$

$$L_{mat}=\rho\sum_\xi \int' d^2\mathbf{q}[|\dot{X}_{\mathbf{q},\xi}|^2-\omega_0^2|X_{\mathbf{q},\xi}|^2], \tag{9}$$

$$L_{int}=-\alpha\sum_{m,\xi}\int' d^2\mathbf{q}K_{m,\xi}[A_{\mathbf{q},m}\cdot\dot{X}^*_{\mathbf{q},\xi}+\text{c.c.}]. \tag{10}$$

Here the prime means the integration over the half of the reciprocal space. Canonically conjugated variables are given as

$$A^*_{\mathbf{q},m} \to P_{\mathbf{q,m}} = \frac{\partial L}{\partial \dot{A}^*_{\mathbf{q},m}} = \epsilon_0 \dot{A}_{\mathbf{q},m}, \qquad (11)$$

$$X^*_{\mathbf{q},\xi} \to Y_{\mathbf{q},\xi} = \frac{\partial L}{\partial \dot{X}^*_{\mathbf{q},\xi}} = \rho \dot{X}_{\mathbf{q},\xi} - \alpha \sum_n K_{n,\xi} A_{\mathbf{q},n}, \qquad (12)$$

and as the complex conjugated relations. As the only difference in comparison to homogeneous dielectrics [5], an infinite number of terms $A_{\mathbf{q},n}$ appears in relation (12). Standard quantization is prescribed by the commutation relations

$$[A_{\mathbf{q},m}, P^*_{\mathbf{q}',n}] = i\hbar \delta_{m,n} \delta(\mathbf{q}-\mathbf{q}'), \qquad (13)$$

$$[X_{\mathbf{q},\xi}, Y^*_{\mathbf{q}',\eta}] = i\hbar \delta_{\xi,\eta} \delta(\mathbf{q}-\mathbf{q}'), \qquad (14)$$

and since now the field variables will be considered as operators. Annihilation operators of the electromagnetic field

$$a_{\mathbf{q},m} = \sqrt{\frac{\epsilon_0}{2\hbar \Omega_m(q)}} \left[\Omega_m(q) A_{\mathbf{q},m} + \frac{i}{\epsilon_0} P_{\mathbf{q},m} \right] \qquad (15)$$

and matter excitations

$$b_{\mathbf{q},\xi} = \sqrt{\frac{\rho}{2\hbar \omega_0}} \left[\omega_0 X_{\mathbf{q},\xi} + \frac{i}{\rho} Y_{\mathbf{q},\xi} \right] \qquad (16)$$

are fulfilling the ordinary boson commutation relations $[a_{\mathbf{q},m}, a^\dagger_{\mathbf{q}',n}] = \delta_{m,n} \delta(\mathbf{q}-\mathbf{q}')$, $[b_{\mathbf{q},\xi}, b^\dagger_{\mathbf{q}',\eta}] = \delta_{\xi,\eta} \delta(\mathbf{q}-\mathbf{q}')$. Extending formally the definitions into the full space of vectors $\mathbf{q}$, the Hamiltonian reads

$$H = \sum_m \int d^2\mathbf{q} \hbar \Omega_m(q) a^\dagger_{\mathbf{q},m} a_{\mathbf{q},m} + \sum_\xi \int d^2\mathbf{q} \hbar \omega_0 b^\dagger_{\mathbf{q},\xi} b_{\mathbf{q},\xi} + \frac{i\hbar}{2} G \sqrt{\omega_0} \sum_{n,\xi} \int d^2\mathbf{q} \frac{K_{n,\xi}}{\sqrt{\Omega_n(q)}} (a_{\mathbf{q},n} + a^\dagger_{-\mathbf{q},n}) \cdot (b^\dagger_{\mathbf{q},\xi} - b_{-\mathbf{q},\xi})$$
$$+ \frac{\hbar}{4} G^2 \sum_{n,m} \int d^2\mathbf{q} \frac{D_{n,m}}{\sqrt{\Omega_m(q) \Omega_n(q)}} (a_{\mathbf{q},n} + a^\dagger_{-\mathbf{q},n}) \cdot (a_{\mathbf{q},m} + a^\dagger_{-\mathbf{q},m}), \qquad (17)$$

the effective interaction constant being $G = \alpha / \sqrt{\epsilon_0 \rho}$. The matrix element is given by

$$D_{n,m} = \sum_\xi K_{n,\xi} K_{m,\xi}$$
$$= \sum_\xi \int_{-l/2}^{l/2} \varphi_n(z) \chi_\xi(z) dz \int_{-l/2}^{l/2} \varphi_m(z') \chi_\xi(z') dz'$$
$$= \int_{-l/2}^{l/2} \varphi_n(z) \varphi_m(z) dz.$$

The general form of polariton transformation diagonalizing the Hamiltonian is given as

$$B_{\mathbf{q},\Omega} = \sum_m W_{\mathbf{q},m} a_{\mathbf{q},m} + \sum_\xi X_{\mathbf{q},\xi} b_{\mathbf{q},\xi} + \sum_m Y_{\mathbf{q},m} a^\dagger_{-\mathbf{q},m}$$
$$+ \sum_\xi Z_{\mathbf{q},\xi} b^\dagger_{-\mathbf{q},\xi}. \qquad (18)$$

Index m exhausts all the cavity modes and ξ similarly does all the modes of the decomposition of matter excitations. The standard diagonalization condition

$$[B_{\mathbf{q},\Omega}, H] = \hbar \Omega B_{\mathbf{q},\Omega} \qquad (19)$$

yields the dispersion relation for eigenfrequency Ω and relations for coefficients in (18). The anticipated operator solution is normalized with respect to the boson commutation relation $[B_{\mathbf{q},\Omega}, B^\dagger_{\mathbf{q}',\Omega'}] = \delta(\mathbf{q}-\mathbf{q}') \delta_{\Omega,\Omega'}$. Straightforward but lengthy calculations lead to the following equations (dependence on $\mathbf{q}$ will be omitted for brevity):

$$(\Omega_m - \Omega) W_m + \frac{1}{2} G^2 \sum_k \frac{D_{m,k}}{\sqrt{\Omega_m \Omega_k}} (W_k - Y_k)$$
$$+ \frac{i}{2} G \frac{\sqrt{\omega_0}}{\sqrt{\Omega_m}} \sum_\xi K_{m,\xi} (X_\xi + Z_\xi) = 0, \qquad (20)$$

$$(\Omega_m + \Omega) Y_m - \frac{1}{2} G^2 \sum_k \frac{D_{m,k}}{\sqrt{\Omega_m \Omega_k}} (W_k - Y_k)$$
$$- \frac{i}{2} G \frac{\sqrt{\omega_0}}{\sqrt{\Omega_m}} \sum_\xi K_{m,\xi} (X_\xi + Z_\xi) = 0, \qquad (21)$$

$$(\Omega - \omega_0) X_\xi + \frac{i}{2} G \sqrt{\omega_0} \sum_m \frac{K_{m,\xi}}{\sqrt{\Omega_m}} (W_m - Y_m) = 0, \qquad (22)$$

$$(\Omega + \omega_0) Z_\xi - \frac{i}{2} G \sqrt{\omega_0} \sum_m \frac{K_{m,\xi}}{\sqrt{\Omega_m}} (W_m - Y_m) = 0. \qquad (23)$$

Finally, the dispersion relation follows as the condition for the existence of the nontrivial solution of linear equations for T_m,

$$T_m = \frac{G^2 \Omega^2}{(\omega_0^2 - \Omega^2)(\Omega_m^2 - \Omega^2)} \sum_n D_{m,n} T_n, \qquad (24)$$

where $T_m = (W_m - Y_m)/\sqrt{\Omega_m}$. The recurrent system of linear equations (24) may be rewritten to the form of a differential equation. Let us define formally the function $A(z) = \Sigma_m T_m \varphi_m(z)$, which is continuous and has continuous derivation dA/dz on the interval $|z| \leqslant L/2$. After simple manipulations the equation for $A(z)$ reads

$$\left[\frac{d^2}{dz^2}+Q_z^2\right]A(z)=-\theta(z)\frac{G^2\Omega^2}{c^2(\omega_0^2-\Omega^2)} \times \int_{-l/2}^{l/2}\sum_{\xi}\chi_{\xi}(z)\chi_{\xi}(z')A(z')dz'. \tag{25}$$

Since the functions $\chi_{\xi}(z)$ are complete on the given interval, the equation (25) coincides with the scalar wave equation (1) for a particular value of susceptibility (6). The function $A(z)$ is therefore given as a Fourier transformation of the classical c-number vector potential

$$A(z)=\frac{1}{2\pi}\int d^2\mathbf{s}\, dt e^{-i\mathbf{qs}-it\Omega}A(\mathbf{r},t) \equiv \sum_m \int dt e^{-it\Omega}A_{\mathbf{q},m}(t)\varphi_m(z).$$

Hence the quantum problem is related to the classical solution of the wave equation with a dispersive lossless inhomogeneous medium. Classical solution yielding a nonorthogonal eigenmode functions is completed by quantum treatment characterized by orthogonal decomposition. A little generalization using the principle of superposition [13] is possible, providing that the electromagnetic field is interacting with a *finite* ensemble of independent oscillators with Lagrangian densities as in (4,5), but distinguished by different parameters $\omega_0,\omega_1,\ldots,\alpha_0,\alpha_1,\ldots,$ and $\rho_0,\rho_1,\ldots$. Then

$$\chi'(\Omega)=\sum_l \frac{G^2(\omega_l)}{\omega_l^2-\Omega^2}, \tag{26}$$

$$\chi''(\Omega)=-\pi\sum_l G^2(\omega_l)\,\delta(\omega_l^2-\Omega^2)$$

and the susceptibility remains lossless.

III. CONCLUSION

Canonical quantization of the electromagnetic field in linear dispersive lossless inhomogeneous media was formulated in terms of the overlapping of wave functions related to quantization of free electromagnetic and matter excitation fields. Waves in a dispersive inhomogeneous medium in classical electrodynamics correspond to generalized polariton transformation. Even if quantum and classical problems yield the same dispersion relations, there is a difference between both the treatments. Since the electromagnetic field itself is not conserved, the respective eigenfunctions are not orthogonal representing the system of quasinormal modes [14]. They may be used for a description of the electromagnetic field inside the cavity; however, since completeness and orthogonality relations should be redefined, the description is more complicated than in the ordinary case of orthogonal modes. This contribution focused the quantum analogy of dispersive inhomogeneity modeled by lossless Lorentz oscillators. The analogous treatment of quantum objects—excitons or dielectrics with losses—needs, however, further analysis.

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