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Entropy of phase measurement: Quantum phase via quadrature measurement

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The content of phase information of an arbitrary phase-sensitive measurement is evaluated using the maximum likelihood estimation. The phase distribution is characterized by the relative entropy—a nonlinear functional of input quantum state. As an explicit example, the multiple measurement of the quadrature operator is interpreted as quantum phase detection achieving the ultimate resolution predicted by the Fisher information. [S1050-2947(96)06705-4]

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I. INTRODUCTION

There are many approaches addressing the problem of quantum phase measurement nowadays. Besides the purely theoretical phase concepts anticipating the existence of quantum phase as an observable conjugated canonically to the number (or difference number) operator, there are several operational treatments addressing the problem of phase shift measurement within the quantum mechanics. Particularly, there are several methods for deriving the phase information from the phase-sensitive measurement of rotated quadrature operator

$$\hat{X}(\theta)=\frac{1}{\sqrt{2}}[\hat{a}e^{-i\theta}+\hat{a}^{\dagger}e^{i\theta}]. \tag{1}$$

The rotated quadrature eigenstates (variable x) appear with the probability depending on the actual phase of the local oscillator θ

$$p(x,\theta)=|\langle\psi|x\rangle_{\theta}|^{2}. \tag{2}$$

Denoting the controlled phase of the local oscillator as θ, an ordinary balanced homodyne-detection scheme measures the quadrature component $\hat{P}(\theta)\equiv\hat{X}(\theta+\pi/2)$—the electric field strength [1]. The following phase interpretations of this measurement have already been proposed.

(i) So called ''phase (measurement) without phase (states)'' was formulated by Vogel and Schleich (VS) [2]. The method is motivated by the geometrical comparison of quadrature and ideal phase measurements in phase space. The quadrature components rotated by an angle are used to define a phase distribution of a single mode of the radiation field. The probability of finding zero electric field plotted versus the local oscillator phase θ,

$$P_{\mathrm{VS}}(\theta)=|\langle\psi|x=0\rangle_{\theta+\pi/2}|^{2},$$

constitutes the proposed phase distribution on the interval $[0,\pi)$. Particularly for coherent input field with the complex amplitude $\alpha=|\alpha|e^{i\varphi}$ the proposed phase distribution reads

$$P_{\mathrm{VS}}(\theta)\propto\exp[-2|\alpha|^{2}\sin^{2}(\varphi-\theta)]. \tag{3}$$

(ii) The phase-sensitive data (2) resulting from the homodyne detection have been interpreted in a different way by Beck, Smithey, and Raymer [3]. Using the optical homodyne tomography method [4], the density matrix may be reconstructed and represented in the phase space. Particularly, the authors used the representation by Wigner function $W(x,p)$ and linked the phase distribution to the marginal distribution of Wigner function

$$P_{W}(\phi)=\int_{0}^{\infty}r\,dr\,W(x=r\cos\phi,p=r\sin\phi). \tag{4}$$

The resulting phase distribution is then periodic on the interval $[0,2\pi)$. Nevertheless, such an approach suffers from a formal flaw. Since the ''probability distribution'' (4) yields negative values for superposition of coherent states (so called ''Schrödinger-cat-like states'') [5], the corresponding operator measure is not positively defined. The procedure cannot therefore be interpreted as any generalized measurement [6]. To get a physically reasonable interpretation, another distribution function, such as, for example, the Q function, should be used.

(iii) This formulation is very close to the treatment suggested by Noh, Fougères, and Mandel (NFM) [7]. In their Scheme 1 two fields are mixed on the beam splitter. The signals detected on the outputs serve for determination of sin and cos functions of phase difference. Provided that the signal is mixed with the strong field of the local oscillator, this measurement corresponds to the simultaneous measurement of $\hat{X}(\theta)$ and $\hat{P}(\theta)$ operators. In this limit the phase distribution coincides with the Shapiro-Wagner (SW) phase

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concept [8] tending to the marginal distribution of the Q function. Assuming the coherent input, the conditional phase distribution of inferred ϕ when θ is true reads

$$P_{\mathrm{SW}}(\phi)=\frac{1}{\pi}\int_0^{\infty} r dr \exp\{-|r-|\alpha|e^{i(\phi-\theta')}|^2\}, \tag{5}$$

where $\theta'=\varphi-\theta$. Hence the given measurement admits many possible interpretations. This ambiguity demonstrates that in addition to the detection scheme, its statistical evaluation should also be optimized.

The purpose of this contribution is to infer the phase information included in arbitrary phase-sensitive data using the maximum likelihood (ML) estimation [6,9]. The phase distribution then yields the ultimate resolution corresponding to the Fisher information if it exists. The proposed method therefore deals with the observed data in the most optimum way. As an explicit example, the quadrature measurement anticipated in all the examples above will be interpreted as quantum phase estimation.

II. PHASE ESTIMATION

Let us formulate the problem for an arbitrary multiple measurement of a discrete phase-sensitive observable [10]. The case of a quantum observable with continuous spectrum will be obtained by a straightforward limiting procedure. Assume the quantum measurement of quantum variable $\hat{Y}$ yielding discrete spectrum $|y_k\rangle$ enumerated for brevity by an index k. The purpose of phase detection is to determine the nonrandom c-number displacement parameter θ in the given interval entering the phase displacement transformation [6] of the quantum state as $|\psi(\theta)\rangle=e^{-i\theta\hat{N}}|\psi\rangle$, $\hat{N}$ being a Hermitian operator. The variable θ represents the *true value* of the phase shift. The estimation on the interval $\theta\in[0,2\pi)$ will be considered for concreteness. The probability of finding the complex amplitude y_k by performing the measurement in transformed quantum state $|\psi(\theta)\rangle$ is given by quantum mechanics as

$$p(y_k,\theta)=|\langle\psi|e^{i\theta\hat{N}}|y_k\rangle|^2.$$

Knowing how all these probabilities depend on the induced phase shift, an unknown phase shift may be inferred on the basis of multiple output data $y_1,y_2,\ldots,y_n$. The phase estimation corresponding to the registered data is given as the phase maximizing the likelihood function

$$L(\phi)=p(y_1,\phi)p(y_2,\phi)\cdots p(y_n,\phi).$$

The common envelope of all the phase histograms obtained by repeating the multiple measurement may be expressed as the conditional phase distribution of inferred phase shift ϕ when θ is true [10],

$$P_{\mathrm{ML}}(\phi|\theta)=\frac{1}{C_n(\theta)}\left\{\prod_k [p_k(\phi)]^{p_k(\theta)}\right\}^n, \tag{6}$$

$p_k(\theta)\equiv p(y_k,\theta)$. The normalization is $C_n(\theta)=\int_0^{2\pi}d\phi\{\Pi_k[p_k(\phi)]^{p_k(\theta)}\}^n$ and index k exhausts all the possible values appearing with nonzero probability. The number of samples n is assumed to be sufficiently large in order to get statistically significant sampling. The distribution may be expressed using the relative entropy

$$S(\phi|\theta)=-\sum_k p_k(\theta)\ln p_k(\phi) \tag{7}$$

as $P_{\mathrm{ML}}(\phi|\theta)\propto e^{-nS(\phi|\theta)}$. The case of phase-sensitive observables with continuous spectrum y may be easily incorporated in this step, defining the relative entropy as

$$S(\phi|\theta)=-\int dy\, p(y,\theta)\ln p(y,\phi). \tag{8}$$

The preferred phase shift is given by the true value θ, since the relative entropy has a minimum at $S(\phi=\theta|\theta)$ due to the Gibbs inequality [11] $S(\phi|\theta)\geqslant S(\phi=\theta|\theta)$. The estimation may be sometimes well approximated by the Gaussian distribution [12] with the variance predicted by the Fisher information. Using the Taylor decomposition of $\ln p_k(\phi)$ at the point $\phi=\theta$ the relative entropy (7) reads

$$S(\phi|\theta)\approx-\sum_k\left(p_k(\theta)\ln p_k(\theta) - \frac{1}{2}\frac{[p_k'(\theta)]^2}{p_k(\theta)}[\phi-\theta]^2+\cdots\right).$$

The prime denotes the derivative $p_k'(\theta)=dp_k(\phi)/d\phi|_{\phi=\theta}$. The first term represents the Shannon entropy $S(\theta)=-\Sigma_k p_k(\theta)\ln p_k(\theta)$, whereas the second one is the Fisher information

$$I(\theta)=\sum_k [p_k'(\theta)]^2/p_k(\theta).$$

The variance of phase distribution in this approximation is simply $\Delta\phi=1/\sqrt{nI(\theta)}$. Provided that the Gaussian approximation cannot be used, the phase resolution may always be evaluated using dispersion

$$D(\theta)=\sqrt{1-|\langle e^{i\phi}\rangle|^2}$$

corresponding for sharp measurements to the ordinary notion of variance $D\approx\Delta\phi$ restricted to the finite interval [6,13].

As an explicit example assume now the quantum measurement of phase-sensitive quadrature component (1) performed for concreteness in the coherent state with the complex amplitude $\alpha=|\alpha|e^{i\varphi}$. The phase shift of the single-mode field is generated by the photon-number operator $\hat{N}=\hat{a}^{\dagger}\hat{a}$. The probability of finding the value x of rotated quadrature operator (1) may be specified for the given signal state as

$$p(x,\theta')=\frac{1}{\sqrt{\pi}}\exp\{-[x-\sqrt{2}|\alpha|\cos\theta']^2\}, \tag{9}$$

where $\theta'=\varphi-\theta$ is the *phase difference* between local oscillator and signal fields. The quantum estimation problem is the following: The distribution (9) is explicitly known as a function of quadrature phase difference θ' and quadrature component x, since these dependencies are always experimentally measurable. The particular choice of Gaussian distribution represents an easy example consistent with the assumptions of Refs. [2,3,7]. Using this knowledge, an *a priori* unknown fixed phase difference should be inferred as accurately as possible on the basis of the limited number of measured data $x_1, x_2, \ldots, x_n$. The corresponding likelihood function may be found as

$$L(\phi) \propto \exp\{-n[\overline{x} - \sqrt{2}|\alpha|\cos\phi]^2\}, \tag{10}$$

where $\overline{x} = \Sigma_{i=1}^{n} x_i/n$. The phase estimation is given as $\cos\phi_{\mathrm{ML}} = \overline{x}/\sqrt{2}|\alpha|$. Repeating this measurement and estimation, the determined phase shift fluctuates in accordance with the prediction (6). The straightforward application of the theory yields the relative entropy as

$$S(\phi|\theta') = \frac{1}{2}\ln\pi + \int_{-\infty}^{\infty} dx p(x,\theta')[x - \sqrt{2}|\alpha|\cos\phi]^2$$

$$= \frac{1}{2}\ln\pi + \frac{1}{2} + 2|\alpha|^2[\cos\phi - \cos\theta']^2,$$

where ϕ is the estimated (inferred) phase difference. The phase distribution inferred after n trials then reads

$$P_{\mathrm{ML}}^{X}(\phi|\theta') \propto \exp\{-2n|\alpha|^2[\cos\phi - \cos\theta']^2\}. \tag{11}$$

This expression is crucial for further considerations and will be detailed in the following. The inferred phase distribution is not shift invariant, i.e., dependent on the difference $\phi - \theta'$. Since the phase distribution (11) depends on $\cos\phi$ function only, it exhibits the mirror symmetry $P_{\mathrm{ML}}(\phi|\theta') = P_{\mathrm{ML}}(2\pi - \phi|\theta')$. Hence the distribution yields twofold ambiguity on the $(0,2\pi)$ interval and this is the immanent part of the method. Nevertheless, this ambiguity does not cause any serious problems and there are several ways to treat it. One may *restrict* the estimation on the half-width phase interval $[0,\pi]$, which effectively tends to the ''phase without phase'' treatment [2] addressed in the following subsection. Alternatively, the quadrature distribution may be recorded for *various* phase angles. These phase-sensitive data may serve for quantum state tomography, or, provided that only quadrature measurements at θ and $\theta + \pi/2$ are available, for evaluation corresponding to the NFM scheme [7] addressed in the next subsection. This method estimates the phase shift on the full interval of the length $[0,2\pi)$.

A. ''Phase without phase''

Provided that statistics of quadrature operator (1) is registered at an unknown phase shift θ' only, the inferred phase distribution is given as (11). The phase measurement yields the one-peak distribution on the interval $[0,2\pi)$ only if $\theta' = 0$ or $\theta' = \pi$. These two possibilities are of course distinguishable by the sign of the measured quadrature components x_i, as the probability distribution (9) indicates. Unfortunately, the phase measurement near the points $\theta' = 0$ or $\theta' = \pi$ yields rather bad resolution, as will be seen in the following. In all the other cases of phase differences θ' the inferred phase distribution (11) does not distinguish between

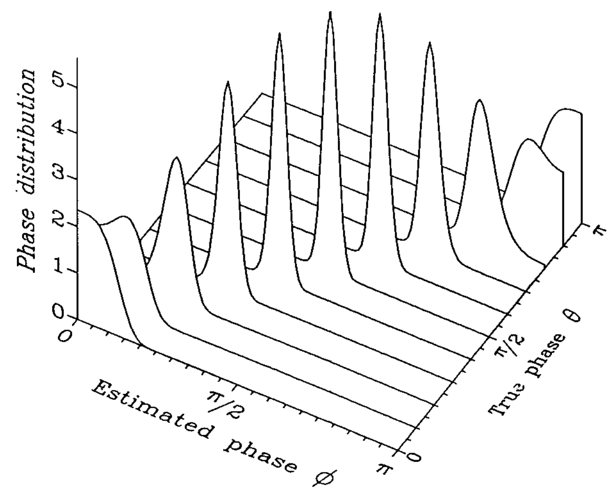

FIG. 1. Phase distribution as function of inferred phase shift ϕ dependent on the true phase shift θ for coherent input with total energy $n|\alpha|^2 = 100$.

the values θ' and $2\pi - \theta'$. This ambiguity may be avoided by estimating the phase difference on the half-width interval $[0,\pi]$ only. The inferred phase distribution dependent on the true phase shift θ is plotted in Fig. 1 for the input coherent field with the real amplitude $(\varphi = 0)$. The estimated phase shift is always localized around the true value, but in general the phase estimation is biased. Assuming sufficiently high energy, the bias may be neglected, but significantly, the accuracy of the estimation depends strongly on the true phase shift. The phase information is sharpest near the point $\theta' = \pi/2$, yielding the limit of coherent state interferometry $\Delta\phi|_{\theta'=\pi/2} \propto 1/\sqrt{n}|\alpha|$. This statistical analysis corresponds well to the semiclassical (linear) approximation, when the phase resolution is predicted by the intrinsic fluctuations of the signal as $\Delta\theta \propto 1/|\alpha||\sin\theta'|$. It represents good estimation in the regime of the best resolution, nevertheless it fails at the points close to $\theta' = 0$. Here also the Fisher information tends to zero, since the quadratic term in the relative entropy disappears. The necessary assumptions concerning the existence of the Fisher information are not fulfilled and, for example, the Cramér-Rao bound is not valid [14]. Nevertheless, the maximum likelihood estimation does not fail [15]. The distribution (11) yields the phase resolution

$$\Delta\phi|_{\theta'=0} \propto \sqrt{\Delta\phi|_{\theta'=\pi/2}}$$

only, which is considerably worse than the resolution at the optimum point. The block diagram of the phase detection based on the maximum likelihood estimation is sketched in Fig. 2. The balanced homodyne detection measures the statistics of the quadrature operator $\hat{P}(\theta)$. Similarly to the previous case of the $\hat{X}(\theta)$ component, the phase difference may be estimated with the conditional phase distribution

$$P_{\mathrm{ML}}^{P}(\phi|\theta') \propto \exp\{-2n|\alpha|^2[\sin\phi - \sin\theta']^2\}. \tag{12}$$

The predicted phase resolution is as in Fig. 1, but shifted by the value $\pi/2$ in both the true and inferred phases. The best resolution is then achieved if $\theta' = 0$. Assuming further the total energy needed for such a realization of multiple mea-

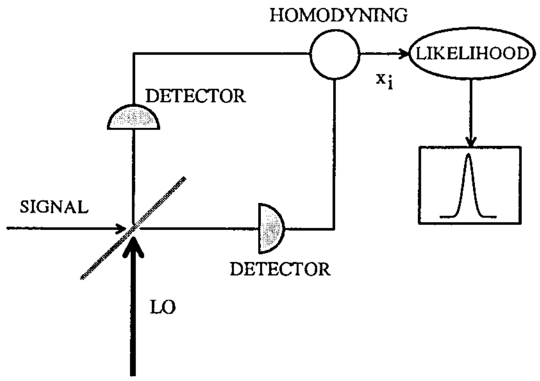

FIG. 2. Scheme of homodyne detection used for phase difference estimation, LO denoting local oscillator.

surement as $n|\alpha|^2$, the maximum likelihood estimation coincides with the proposal of Vogel and Schleich [2]. Estimation theory therefore naturally extends the ''phase without phase'' concept.

B. Phase estimation in the NFM scheme

Maximum likelihood estimation tends also to the very natural interpretation of the phase-sensitive measurement of Noh, Fougères, and Mandel mentioned as example (iii) for motivation. The measurement according to Scheme 1 [7] for quantum signal and strong (classical) field of the local oscillator may be interpreted as simultaneous registration of quadrature operators $\hat{X}$ at angles θ and $\theta+\pi/2$. Hence the phase distribution inferred after multiple measurement is given as the product of the distributions inferred from the measurement of $\hat{X}$, and those inferred from detection of the $\hat{P}$ operator. The former is given by relation (11) whereas the latter is given by (12). The resulting phase distribution is then given by the normal distribution on the circle (von Mises) [13],

$$P_{\mathrm{ML}}(\phi|\theta')=\frac{1}{2\pi I_0(\kappa)}\exp[\kappa\cos(\phi-\theta')], \qquad (13)$$

where $\kappa=4n|\alpha|^2$. Significantly, this measurement is unbiased, centered at θ', and characterized by the dispersion

$$D^2=1-[I_1(\kappa)/I_0(\kappa)]^2,$$

$I_0(\kappa),I_1(\kappa)$ being the modified Bessel functions. The resolution is no longer phase shift dependent since the phase fluctuations inferred from the quadrature components $\hat{X}$ and $\hat{P}$ are complementary. This interpretation may be compared with the standard quantum mechanical treatment represented by the Shapiro-Wagner phase concept (5), where the phase shift is inferred after each measurement separately without any accumulation of information. The phase distribution may be approximated by the normal distribution for sufficiently high energy only. The difference between ML and standard quantum mechanical (SW) estimations may be simply demonstrated on the evaluation of multiple data $(x_1,p_1),\ldots,(x_n,p_n)$. The likelihood function may be found analogously to the relation (10) as

$$L(\phi)\propto\exp\{\kappa\cos(\phi-\phi_{\mathrm{ML}})\},$$

where

$$e^{i\phi_{\mathrm{ML}}}=(\overline{x}+i\overline{p})/\sqrt{\overline{x}^2+\overline{p}^2},$$

$\overline{x}=\Sigma_{j=1}^n x_j/n$, and $\overline{p}=\Sigma_{j=1}^n p_j/n$. On the other hand, each detected pair (x_j,p_j) may be immediately interpreted as registration of phase shift

$$e^{i\phi_{\mathrm{SW}}}=(x_j+ip_j)/\sqrt{x_j{}^2+p_j{}^2}$$

for any $j=1,\ldots,n$. The common envelopes of histograms of the values ϕ_{ML} and ϕ_{SW} are given as the predictions (13) and (5), respectively. Analogously, the ML analysis might be applied to the estimation of phase using the phase-sensitive data resulting from the quadrature measurements at phase shifts $\theta'+\Delta_i$, Δ_i being controlled (known) values of phase shift of the local oscillator. The ML analysis of these multiple data, which may also serve for quantum state tomography, is beyond the scope of this contribution.

There is still another interpretational point that should be mentioned. The phase distribution discussed in this contribution is the conditional distribution describing how the estimated phase is spread around the true value. This might be contrasted to the genuine phase distribution of the phase *per se* resulting from some ideal phase concepts. The viewpoint advocated here is motivated by pragmatic interpretation. Assuming quantum theory as theory of measurement, observable quantities only are of interest. Since both the quantum state and detection method are inseparable in affecting the result, it does not seem to be reasonable to distinguish between the effect (phase itself) and its quantum ''measurement.'' Any ideal phase concept represents from this viewpoint some special choice of estimation procedure only. Hence there is no discrepancy between quantum mechanics and mathematical statistics. This was explicitly demonstrated using the example of simultaneous measurement of quadrature operators: The evaluation of phase just after the registration of unnormalized $\sin\phi$ and $\cos\phi$ is in accordance with the standard approaches used in quantum mechanics. On the other hand, the optimum strategy based on ML estimation averages the values of trigonometric fuctions and then evaluates the phase shift.

III. CONCLUSION

We demonstrated that any phase-sensitive measurement may serve for statistical prediction of phase shift. The content of phase information may be evaluated using the relative entropy of the phase depending on the observable probabilities only. The resolution predicted by the Fisher information is then achieved if it exists. The proposed method based on the maximum likelihood estimation uses the information accumulated in the process of multiple measurement in the optimum way. This treatment suits the experimental conditions better than sophisticated phase concepts. Particularly, the phase distribution inferred on the basis of a given measurement should be rather associated with nonlinear functionals (likelihood functional, relative entropy) than with the linear ones such as the distribution functions on the phase space are (Wigner function, Q function). In realistic experiments the detailed statistical analysis free of any *a priori* restricting assumptions is always necessary, since standard mathematical assumptions need not always be fulfilled. This was demonstrated here using the specific example of the coherent state, where the Gaussian distribution provides zero Fisher measure. Even if the relation between quadrature operator and phase is detailed here, it does not mean that the registration of the quadrature operator is the best way to determine the phase shift. As is well known, interferometers better serve this purpose, but this topic will be addressed elsewhere.

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