Extremal properties of near-photon-number eigenstate fields

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The near-photon-number eigenstates introduced by Yuen [Phys. Rev. Lett. **56**, 2176 (1986)] link together squeezing, antibunching, and phase properties of light. They are the crescent states generated by the crescent operator introduced here, and physically they represent another explicit example of minimum uncertainty states of the Shapiro-Wagner phase concept.

I. INTRODUCTION

There is strong interest in the improvement of the performance of optical devices beyond the shot-noise limit predicted by classical optics. The focus of quantum optics is therefore on the nonclassical behavior of light, such as squeezing [1], antibunching, and sub-Poissonian photocount statistics [2]. An "old" problem of the phase operator in quantum mechanics has also received renewed and increasing interest, since many measurable effects may be associated just with the phase shift of a single-mode field.

The purpose of this Brief Report is to emphasize that the nonclassical properties of light are simply associated with the phase properties in the feasible model of Shapiro-Wagner (SW) phase measurement via simultaneous registration of quadrature components [3–5]. The states specified in Ref. [6] as noise minimum states (NMS), which minimize the variance of photon number Δn under the given constraints, provide the ultimate phase resolution of SW phase measurement predicted in Ref. [7]. The extremal properties of an example of NMS introduced by Yuen [8] as a near-photon-number eigenstate field are specified here. They are generated by the state reduction during the parametric down conversion of four-wave mixing [9] and provide the crescent shape of Q—quasidistribution typical for the Susskind-Glogower (SG) phase concept [10]. This feature may be expressed easily by the action of the crescent operator. Moreover, such states are also stationary states in a medium with Kerr nonlinearity and they can be treated as displaced Fock states [11]. As the last interesting property, let us note that they are the eigenstates of the operator, associated with the simultaneous measurement of photon number and quadrature components. The similar combined conditional probability distributions are investigated in connection with superposition of classically distinguishable quantum states [12]. All these reasons support the importance of investigated states in quantum optics.

II. CRESCENT STATES

The following inequality was derived recently [6]:

$$\Delta n \geq |\bar{a}| \frac{\lambda}{\lambda_1 \lambda_2} , \tag{1}$$

where

$$(\lambda_{1,2})^2 = 1 + 2[\langle \mathrm{var}\hat{a}^\dagger \, \mathrm{var}\hat{a} \rangle \pm |\langle (\mathrm{var}\hat{a})^2 \rangle|]$$

are half-axes of the error ellipse, $\lambda^2 = \lambda_1^2 \sin^2\Phi + \lambda_2^2 \cos^2\Phi$, and $\Phi = \frac{1}{2}\arg\langle(\mathrm{var}\hat{a})^2\rangle - \arg\bar{a}$. In the following we will abbreviate using the symbols var for variance and Δ for the root-mean-square of variance of an appropriate operator, for convenience. Expression (1) was derived on the basis of the commutation relation

$$[\hat{n}, \hat{X}_1(\theta)] = -i\hat{X}_2(\theta)$$

and consequently

$$\Delta n \geq \frac{|\langle \hat{X}_2(\theta) \rangle|}{2\Delta X_1(\theta)} .$$

The phase-dependent quadrature operators $\hat{X}_{1,2}$ are given as

$$\hat{X}_1 = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}) \quad \hat{X}_2 = -i(\hat{a}e^{-i\theta} - \hat{a}^\dagger e^{i\theta}) .$$

Relation (1) can be obtained by taking the maximum over θ on the right-hand side. Inequality (1) then formally makes it possible to introduce a quantity like phase variance fulfilling the uncertainty relation with the photon number variance as

$$\Delta\phi = \frac{\lambda_1 \lambda_2}{2|\bar{a}|\lambda} . \tag{2}$$

The explicit calculations [4,7] confirm this conclusion, apart from the multiplicative factor $\sqrt{2}$, for strong two-photon coherent and coherent fields, since the value of $\sqrt{2}\Delta\phi$ represents the smallest possible phase resolution in the Shapiro-Wagner phase concept, if both quadrature operators $\hat{X}_1(\theta)$ and $\hat{X}_2(\theta)$ are measured simultaneously.

Now we will clear up for which states the equality in inequality (1) holds exactly. These states are the minimum uncertainty states (MUS) of the photon-number operator $\hat{n}$ and the quadrature operator, since they resemble the number-phase MUS of the SG phase operator investigated in Refs. [13] and [14]. Their mathematical development goes analogically and they represent another special example of NMS, mentioned in Ref. [6] as a polynomial solution. The equality sign in the inequality (1) holds for states that diagonalize the non-Hermitian operator,

$$[\hat{n}-i|\xi|\hat{X}_1(\theta)]|\psi\rangle=\Omega|\psi\rangle \ , \tag{3}$$

in analogy to the analytical approach of Ref. [14]; the complex parameter ξ is advantageously defined as $\xi=i\exp(i\theta)|\xi|$. The properties of these extremal states will be considered here.

The value of Δn can be found directly without tedious calculations if we notice that the expression on the right-hand side of inequality (1) represents the maximum over θ. Then

$$(\Delta n)_{\text{MUS}}=\frac{|\langle \hat{X}_2(\theta)\rangle|}{2\Delta X_1(\theta)}\le|\bar{a}|\frac{\lambda}{\lambda_1\lambda_2} \ ,$$

and, since inequality (1) takes place for all states, the following equality is therefore valid:

$$(\Delta n)_{\text{MUS}}=|\bar{a}|\frac{\lambda}{\lambda_1\lambda_2} \ . \tag{4}$$

To obtain the explicit solution for photon-number and quadrature MUS we rewrite Eq. (3) advantageously to the form with displaced annihilation operator $\hat{A}=\hat{a}-\xi$,

$$[\hat{A}^\dagger\hat{A}+2\xi^*\hat{A}]|\psi\rangle=(\Omega-|\xi|^2)|\psi\rangle \ . \tag{5}$$

The solution can be found in the form of decomposition, $|\psi\rangle=\sum_n a_n|n\rangle$, respective to the displaced Fock states generated by the $\hat{A}$ operator. Then it is necessary to fulfil the recursion relation

$$(n-\Omega+|\xi|^2)a_n=-2\xi^*(n+1)^{1/2}a_{n+1} \ , \quad n\ge 0 \ . \tag{6}$$

To converge the decomposition requires the cutoff in the form

$$\Omega-|\xi|^2=M \ , \quad M=0,1,2,\ldots,a_k=0 \quad \text{for } k\ge M+1$$

and

$$a_k=a_0(k!)^{1/2}\begin{bmatrix} M \\ k\end{bmatrix}(2\xi^*)^{-k} \quad \text{for } k=0,\ldots,M \ .$$

The solution may be expressed in the closed form as

$$|\psi\rangle=[a_0/(2)^M](1+\hat{a}^\dagger/\xi^*)^M|\xi\rangle_{\text{coh}} \ , \tag{7}$$

or alternatively, in the basis of Fock states using the rule for expanding the generating function of the generalized Laguerre polynomial [15] L_M^k as

$$\begin{aligned}|\psi\rangle&=[a_0\exp(-|\xi|^2/2)/(2^M)]\\&\quad\times(1+\hat{a}^\dagger/\xi^*)^M\exp(\xi\hat{a}^\dagger)|0\rangle\\&=[a_0\exp(-|\xi|^2/2)/(2^M M!)]\\&\quad\times\sum_{m=0}^{\infty}(\xi^*)^{-m}(m!)^{1/2}L_M^{M-m}(-|\xi|^2)|m\rangle \ .\end{aligned} \tag{8}$$

Using the identity

$$\exp(|\xi|^2)(\hat{a}^\dagger+\xi^*)^M|\xi\rangle=(M!)^{1/2}\exp[|\xi|\hat{X}_2(\theta)]|M\rangle \ , \tag{9}$$

the state (7) may also be treated, analogous to the coherent states [16], as nonunitarily-shifted Fock states $|M\rangle$ (Ref. [11]),

$$|\psi\rangle=N\exp[|\xi|\hat{X}_2(\theta)]|M\rangle \ , \tag{10}$$

where the normalization N is given as

$$\begin{aligned}N(\xi,M)&=\langle M|\exp[2|\xi|\hat{X}_2(\theta)]|M\rangle^{-1/2}\\&=(M!)^{1/2}\exp(-|\xi|^2)|a_0|/(2|\xi|)^M &\text{(11a)}\\&=\exp(-|\xi|^2)[M!/L_M(-4|\xi|^2)]^{1/2} \ . &\text{(11b)}\end{aligned}$$

Equality (11a) follows from identity (9), whereas expression (11b) is obtained using decomposition (8) of the extremal state with the argument 2ξ.

The spectrum of the operator equation (3) is real and therefore we can derive easily that $\bar{X}_1(\theta)=0$ (Ref. [17]); $\bar{X}_2(\theta)=2|\bar{a}|$, $\bar{n}=M+|\xi|^2$. Other moments important for systematic investigation of these extremal states in the spirit of Ref. [6] can be developed as

$$\bar{a}=\xi[1+2L_M^1(-4|\xi|^2)/L_M(-4|\xi|^2)] \ ,$$

$$\Delta X_1(\theta)=\left|\frac{\bar{a}}{\xi}\right|^{1/2} \ ,$$

and

$$\Delta n=(|\xi|\,|\bar{a}|)^{1/2} \ .$$

The sub-Poissonian nature of such light is evident, since $|\bar{a}|\ge|\xi|$ and therefore $\Delta n\le|\bar{a}|\le(\bar{n})^{1/2}$. For completeness we will set up the appropriate spheroidal equation associated with inequality (1) in the form

$$[\text{var}\hat{n}+i|\xi|\text{var}\hat{X}_1(\theta)]\times[\text{var}\hat{n}-i|\xi|\text{var}\hat{X}_1(\theta)]|\psi\rangle=0 \ ,$$

or more apparently

$$[\hat{n}^2-2M\hat{n}-\xi^{*2}\hat{a}^2-\xi^2\hat{a}^{\dagger 2}-\xi^*\hat{a}-\xi\hat{a}^\dagger+|\xi|^2+(M+|\xi|^2)^2]|\psi\rangle=0 \ . \tag{12}$$

The extremal states therefore minimize the photon-number variance under the constraints of given amplitude and variance of one quadrature component.

The antinormally ordered quasidistribution $Q(\alpha)$

$$Q(\alpha)=|\langle\alpha|\psi\rangle|^2/\pi\sim|\alpha+\xi|^{2M}\exp(-|\alpha-\xi|^2) \tag{13}$$

yields the typical crescent shape, as is seen in Fig. 1. We suggest, therefore, introducing formally the operator of

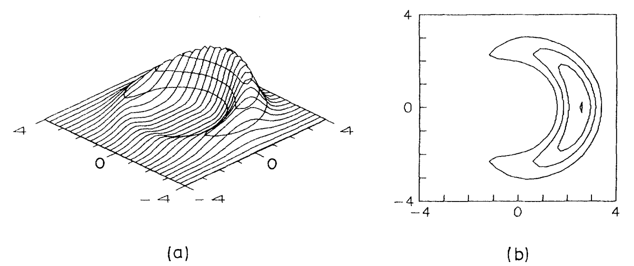

FIG. 1. (a) The plot and (b) the view on the Q representation of the near-photon-number eigenstate for $M=6$, $\xi=0.1$. The contours are at the levels 0.001, 0.002, 0.003, and 0.004. The x and y axes lie in Reα and Imα, respectively.

crescent shape $\hat{C}(\xi)$,

$$\hat{C}(\xi)=\exp[|\xi|\hat{X}_2(\theta)] , \tag{14}$$

as the complex extension of displacement operator $\hat{D}(\alpha)$ representing the nonunitary transformation. The state given by expression (10) is then the crescent state. The crescent operator plays the same role for probability distributions of photon number and phase as the operator of squeezing does for probability distributions of quadrature operators. The properties of both operators are therefore analogical and the crescent operator is relevant to the phase measurement in the Shapiro-Wagner phase concept. A similar behavior, but without such an easy formalism, can be found also in other phase concepts [10].

III. NEAR-PHOTON-NUMBER EIGENSTATE FIELDS

The crescent transformation is associated with the generalized quantum measurement [18] of photon number and both the quadrature components, as the real and imaginary parts of diagonalized operator $\hat{Y}$ in Eq. (3),

$$\hat{Y}=\hat{n}-i|\xi|\hat{X}_1(\theta) ,$$

indicate. This may be verified explicitly considering the state reduction for correlated photons in nonlinear optical processes [8,9]. We will show this, adopting the results of Ref. [9]. Let us suppose that the nonlinear coupling device performs the SU(1,1) transformation of input modes $\hat{a}_{1,2}$ into the output modes $\hat{b}_{1,2}$ according to the rules

$$\begin{aligned}\hat{b}_1&=\hat{a}_1\cosh\tau+a_2^{\dagger}e^{-i\delta}\sinh\tau ,\\ \hat{b}_2&=\hat{a}_2\cosh\tau+\hat{a}_1^{\dagger}e^{-i\delta}\sinh\tau ,\end{aligned} \tag{15}$$

providing the input state in coherent states α_0 (respectively, β_0). If one input mode (port 1) is measured and found in the pure photon-number state $|m\rangle$, then the field on the other port 2 is prepared in the state with the reduced Q_m quasidistribution,

$$\begin{aligned}Q_m(\alpha)&\sim P_m(\alpha)\equiv {}_1\langle m|_2\langle\alpha|\hat{\rho}|\alpha\rangle_2|m\rangle_1/\pi\\ V&\equiv 1/(\pi m!\cosh^2\tau)(\tanh\tau)^{2m}|\alpha-\bar{\alpha}|^{2m}\exp[-|\alpha-\alpha_0|^2/\cosh^2\tau-|\alpha-\bar{\alpha}|^2\tanh^2\tau] ,\end{aligned} \tag{16}$$

where $\bar{\alpha}=\alpha_0+e^{i\delta}\beta_0^*/\tanh\tau$. Such states evidently may be treated as the unnormalized states

$$|\chi\rangle\sim(\hat{a}^{\dagger}-\mu^*)^m|\nu\rangle_{\rm coh} , \tag{17}$$

where the parameters are given as $\mu=\alpha_0+e^{i\delta}\beta_0^*/\tanh\tau$ and $\nu=\alpha_0+e^{i\delta}\beta_0^*\tanh\tau$. The states (17) may be normalized easily and parametrized using the displacement and crescent operators as

$$\begin{aligned}|\varphi,(\xi,m)\rangle&=N(\xi,m)\hat{D}(\varphi)\hat{C}(\xi)|m\rangle ,\\ Q_m(\alpha)&=|\langle\alpha|\varphi,(\xi,m)\rangle|^2/\pi ,\end{aligned} \tag{18}$$

$N(\xi,m)$ is the normalization given in expression (11b) and

$$\begin{aligned}\varphi&=(\mu+\nu)/2=\alpha_0+e^{i\delta}\beta_0^*/\tanh2\tau ,\\ \xi&=(\nu-\mu)/2=-e^{i\delta}\beta_0^*/\sinh2\tau .\end{aligned} \tag{19}$$

This state reduces to the crescent states without displacement, if the condition $\varphi=\alpha_0+e^{i\delta}\beta_0^*/\tanh2\tau=0$ is achieved.

The considered near-photon-number eigenstates will play an important role in more sophisticated detection techniques in the future; the combined conditional probability distributions investigated in Ref. [12] are such an example. The states (18) may be associated with the probability operator measure (POM) respective to the simultaneous measurement of photon number and quadrature components, as is done in the Appendix.

IV. CONCLUSIONS

The investigated states represent the extremal states respective to the measurable effects of phase, squeezing, and antibunching. They exhibit analogical features comparing to the number-phase MUS of SG phase concepts, but the model considered here seems to be more simple and effective. In particular, the operator performing the crescent shape transformation may be introduced simply. The near-photon-number eigenstates are suited to the above-mentioned measurements simultaneously, and therefore the improvement of performance by some combined measurements may be expected in the future.

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APPENDIX: POM FOR SIMULTANEOUS DETECTION OF PHOTON NUMBER AND BOTH THE QUADRATURE COMPONENTS

The considered probability distribution (16) can be treated as a result of generalized quantum measurement on output modes of the SU(1,1) coupling device, since

$$P_m(\alpha)=\mathrm{Tr}[\hat{R}_m(\alpha)\hat{\rho}_{\mathrm{out}}] \ . \tag{A1}$$

The POM is given by the operators

$$\hat{R}_{m,\mathrm{out}}(\alpha)=(1/\pi)|m\rangle_{1\,1}\langle m|\otimes|\alpha\rangle_{2\,2}\langle\alpha| \ , \tag{A2}$$

$|m\rangle$ being the Fock state and $|\alpha\rangle$ being the coherent state of the output modes $\hat{b}_{1,2}$ of the SU(1,1) coupling device (15), which performs the transformation of signal input mode $\hat{a}_1$ and of some auxiliary (image) input mode $\hat{a}_2$. Our task is to find, for a given image input field, the POM $\hat{R}_{m,\mathrm{in}}(\alpha)$ on the signal Hilbert space. The physical specification is therefore the following: the input mode 1 is in an arbitrary state with the density matrix represented as

$$\hat{\rho}_{\mathrm{in}}=\int d^2\alpha_0 P(\alpha_0)|\alpha_0\rangle\langle\alpha_0| \ , \tag{A3}$$

whereas the image input field 2 remains, as in Sec. III, in the coherent state β_0. The formal solution

$$\hat{R}_{m,\mathrm{in}}(\alpha)\equiv{}_{2,\mathrm{in}}\langle\beta_0|\hat{R}_{m,\mathrm{out}}(\alpha)|\beta_0\rangle_{2,\mathrm{in}} \tag{A4}$$

can be expressed using the projectors into the displaced crescent states (18), if we assume that

$$P_m(\alpha)=\int d^2\alpha_0 P(\alpha_0)|\langle\alpha|\varphi,(\xi,m)\rangle|^2 q_m \tag{A5}$$

$$=\int d^2\alpha_0 P(\alpha_0)|\langle\alpha_0|\overline{\varphi},(\xi,m)\rangle|^2 q_m \tag{A6}$$

$$\equiv\mathrm{Tr}[\hat{\rho}_{\mathrm{in}}\hat{R}_{m,\mathrm{in}}(\alpha)] \ , \tag{A7}$$

where

$$\hat{R}_{m,\mathrm{in}}(\alpha)=|\overline{\varphi},(\xi,m)\rangle\langle(m,\xi),\overline{\varphi}|q_m/\pi \tag{A8}$$

and $\overline{\varphi}=\alpha+e^{i\delta}\beta_0^*/\tanh 2\tau$ and

$$q_m=(\sinh\tau)^{2m}/(m!\cosh^2\tau)\times\exp(-|\beta_0|^2/\cosh^2\tau)\times L_m(-4|\beta_0|^2/\sinh^2 2\tau) \ .$$

Relation (A5) follows from (A3) and from the knowledge of quasidistribution (16) for an input signal field in the coherent state. The equality in (A6) is valid due to the dependence of the state (18) on the complex amplitude α_0, since the parameters α,α_0 may be changed. The resolution of the identity operator

$$\left[\frac{1}{\pi}\right]\sum_{m=0}^{\infty}\int\int d^2\varphi q_m|\varphi,(\xi,m)\rangle\langle(m,\xi),\varphi|=\hat{1} \tag{A9}$$

is the consequence of expression (A2).

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