Noise minimum states and the squeezing and antibunching of light

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The purpose of this paper is to investigate intrinsic restrictions of quantum mechanics put on the correlation function $g^{(2)}$ by values of lower-order moments, which characterize noise in interferometric measurements. Some general consequences regarding the relation between antibunching and squeezing are obtained. The class of fields with the minimum value of $g^{(2)}$ for given noise is specified and the relation of such states to the Kerr medium is emphasized. The concept of noise minimum states can replace number-phase minimum uncertainty states.

I. INTRODUCTION

To obtain usable information about statistical properties of light, quantum optics often deals with moments of the operator of electromagnetic field $\hat{E}$ only, instead of using the complete information contained in the density matrix $\hat{\rho}$. In such a way, of course, some of the content of the information about light is lost. Nevertheless, dualism of the light quanta is preserved if we use the moments of the field operator up to fourth order.[1,2] However, if we accept this approach, a number of new problems arise from the theoretical point of view. The density matrix $\hat{\rho}$ must obey the following important conditions:[3] (1) $\hat{\rho}$ is normalizable,

$$\mathrm{Tr}(\hat{\rho})=1 , \tag{1.1a}$$

(2) $\hat{\rho}$ is a positive-definite Hermitian operator, and therefore (3) the necessary condition

$$\mathrm{Tr}(\hat{\rho}^2)\leq 1 , \tag{1.1b}$$

where the equality holds for pure states.

Consequently some intrinsic restrictions appear, which apply to all the possible moments of the field operators. They are known as the generalized Cauchy-Schwarz (GCS) inequalities,[3,4] and their origin is the same as that of the violation of Bell inequalities,[5] for example. In other words, they are predictions of quantum mechanics. In the spirit of the second quantization we will deal with the creation and annihilation operators of the single-mode stationary boson field ($\hat{a}^\dagger$ and $\hat{a}$), and the question is: What are the mutual restrictions put on the moments of the operators $\hat{a}^\dagger,\hat{a}$ up to the fourth order? As a special result of this general task we can obtain some consequences for the relation between squeezing and antibunching of light,[6] because the former effect involves the operators up to the second-order,[1] whereas the latter one includes them up to the fourth order.[2] This connection was recently obtained for special states of light. Walls[7] showed that ideal vacuum squeezed states are bunched ($g^{(2)}>1$) and Bondurant and Shapiro[8] found maximal antibunching of ideal squeezed states as $(\Delta n)^2=(\bar{n})^{2/3}$ for mean photon number $\bar{n}$ large, where Δn is the root mean square of $\hat{n}$ (rms). The stronger antibunching $(\Delta n)^2=(\bar{n})^{1/3}$ occurs for the very special Kitagawa-Yamamoto states[9] (amplitude squeezed states), generated with the help of Kerr nonlinearity, and therefore a natural question arises: How closely can we approach the "ideal" case $(\Delta n)^2=0$ under some given conditions? We will try to clarify these questions.

For this purpose we will introduce a new class of quantum states of light-noise minimum states (NMS's). They will be defined as all the states that minimize the rms of $\hat{n}$ under given constraints of lower-order moments. This definition represents an alternative approach to the concept of minimum uncertainty states[10] (MUS's), especially number-phase MUS's when we try to minimize the uncertainty product of two canonically conjugated operators, especially of photon-number and "phase" operators. Both concepts—NMS's and MUS's—when applied to the simplest case of operators up to the second order in $\hat{a}^\dagger,\hat{a}$ and to quadrature operators, respectively, tend to well-known ideal squeezed states (for NMS's this will be shown in Sec. II). Fundamental differences appear when also moment $\hat{n}^2$ is involved. Number-phase MUS's minimize the rms of $\hat{n}$, but nobody knows under which physical circumstances. There are two noncommuting operators $\hat{C}$ and $\hat{S}$ associated with classical phase and therefore "phase" has a problematic meaning in quantum optics, except for strong fields. These difficulties, contained also in the mathematical structure of number-phase MUS's, hinder possible applications. As a matter of fact, the rms of $\hat{n}$ is ranged not only by predictions of values of the rms of $\hat{C}$ and $\hat{S}$, respectively, but also by all measurements of noncommuting operators; for example, using such measurements which include operators $\hat{a},\hat{a}^2,\hat{n}$ (respectively, their adjoint operators) only. This is the main idea of NMS's and it promises better conditions for the interpretation of the minimum value of the rms of $\hat{n}$, because only squeezing is connected with the measurement of such moments. Better conditions for a physical interpretation of NMS's will perhaps increase possible realizations and applications. Nowadays antibunching and squeezing are utilized separately and are also treated as independent effects of different order, but in the future it will perhaps be necessary to use them simultaneously.

The usage of NMS will show to what extent that is possible.

To support the concept of NMS's we will emphasize in Sec. 4 that such states incorporate many interesting quantum states of light that are widely used in the investigations of quantum optics, i.e., superposition states,[11] ideal squeezed states,[7,12] amplitude squeezed states,[9] photon-number states (Fock states), etc. All these states possess the common feature mentioned above, and therefore the concept of NMS's unifies a variety of states in quantum optics.

To avoid confusion it will be necessary to clarify the terminology regarding squeezing. The notion of quadrature noise squeezing is straightforward—for some phase parameter Φ the quadrature variance $\langle [\Delta\,\mathrm{Re}(\hat{a}\,e^{-i\Phi})]^2\rangle$ falls below the semiclassical shot-noise limit of $\frac{1}{4}$, associated with a coherent state.[1] Unfortunately, the literature in the field has not been uniform in its use of the word "squeezing." Yuen's two-photon coherent state[13] (TCS) is a pure state with a Gaussian normally ordered form that is squeezed, and we will distinguish this ideal squeezed state from more general entities that satisfy the preceding squeezing condition that a state that is squeezed need not be a squeezed state, i.e., that it need not be a TCS. The quadrature variance condition given above will be replaced in Sec. II by an equivalent relation [see inequality (2.3) below]. In the following we will use the abbreviation TCS also for ideal squeezed states,[7,12] ordinary squeezed states,[9] etc. (as they differ in the parametrization only[1]) to distinguish them from the states that are only squeezed according to the quadrature variance condition. In the same way, since we suppose an ideal process of detection, sub-Poissonian or antibunched light is the same and we will use the term antibunching.

The paper is organized as follows. The relations between moments based on the GCS inequalities are given in Sec. II. The main results are extended in Sec. III, where the NMS's are mathematically developed. In Sec. IV some special examples of NMS's are concluded, together with considerations about the relation between antibunching and squeezing. Section V contains some other problems related to NMS's.

II. RESTRICTIONS

Our tool will be the GCS inequality[3,4]

$$|\langle \hat{A},\hat{B}\rangle|^2 \le \|\hat{A}\|^2\,\|\hat{B}\|^2\ , \tag{2.1}$$

where $\langle \hat{A},\hat{B}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A}^{\dagger}\hat{B})$, $\|\hat{A}\|^2 = \mathrm{Tr}(\hat{\rho}\hat{A}^{+}\hat{A})$, which is valid for operators with bounded norms. The scalar product allows the possibility that $\|\hat{A}\|=0$ for $\hat{A}\neq 0$. The equality in (2.1) occurs only for the states $\hat{\rho}$, where (i) $\|\hat{A}\|=0$, $\|\hat{B}\|=0$ or (ii) $\|\hat{A}\|\,\|\hat{B}\|\neq 0$ and $(\hat{A}-\lambda\hat{B})\hat{\rho}=0$ for some λ. We would like to emphasize that inequality (2.1) cannot be violated in quantum mechanics in contrast to the classical Cauchy-Schwarz inequality,[5] which can.

Consequences for lower-order moments are quite simple and the following inequality can be written:[6]

$$|\bar{a}|^2 \le \bar{n} \tag{2.2a}$$

and

$$|\overline{a^2}-(\bar{a})^2|^2 \le (\bar{n}-|\bar{a}|^2)^2+\bar{n}-|\bar{a}|^2\ , \tag{2.2b}$$

where the bar above the operators means the mean value and the abbreviation $\hat{a}^{\dagger}\hat{a}=\hat{n}$ is also used. The equality occurs for coherent states in (2.2a) and the TCS's in (2.2b) only. All possible moments must be in accordance with this relation and we can represent them advantageously in a graphical way, if we choose the three-dimensional configuration space with the axes $x+iy=\overline{a^2}$, $z=\bar{n}$ with the complex parameter $\bar{a}$. All moments above the hyperboloid H are then possible [Figs. 1(a) and 1(b)]. Another finer classification of this space according to squeezing is provided by the condition[14]

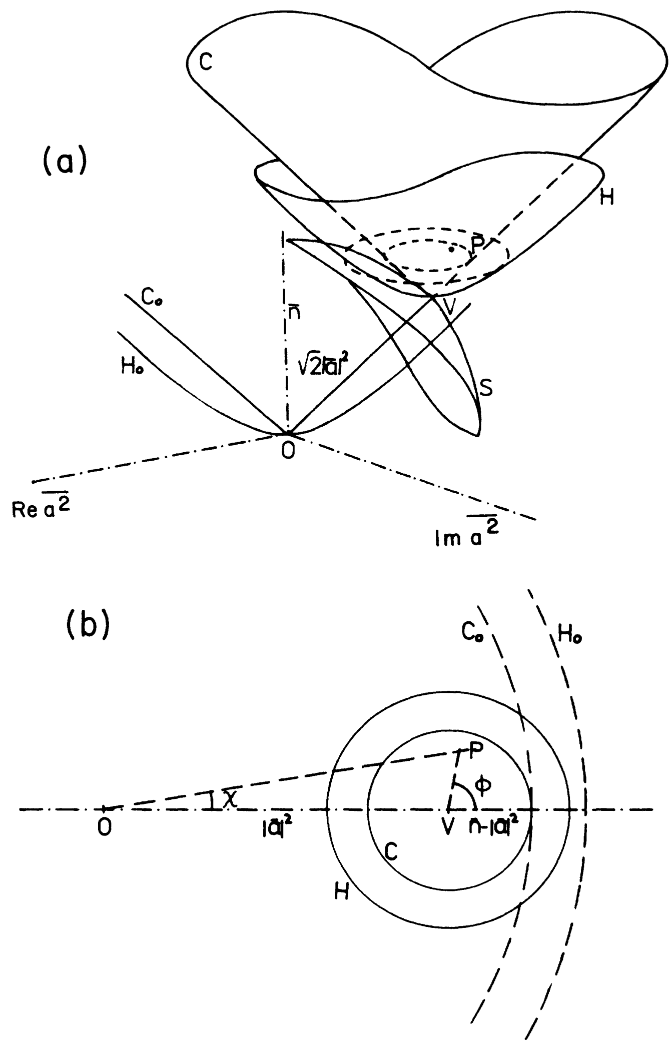

FIG. 1. (a) Three-dimensional subspace of all possible second-order moments for a given value $\bar{a}$ ranged by the rotationally symmetric hyperboloid H. The cone C is the boundary for moments representing squeezed light. H_0 and C_0 have the same meaning for $\bar{a}=0$. All antibunched Gaussian states lie inside the sphere S with center at origin O and radius $\sqrt{2}|\bar{a}|^2$. Vertex V has coordinates $[\mathrm{Re}(\bar{a})^2, \mathrm{Im}(\bar{a})^2, |\bar{a}|^2]$. (b) Upper view of the cut of (a) by the plane $\bar{n}=$const. General points may be represented by the parameters $(\chi, |\overline{a^2}|)$ or alternatively $(\phi, |\overline{a^2}-(\bar{a})^2|)$.

$$|\overline{a^2}-(\bar{a})^2| > \bar{n}-|\bar{a}|^2 \;, \tag{2.3}$$

which represents the definition for squeezing (in general). The geometrical meaning is also simple—all points below the cone C are squeezed and the TCS's provide the maximally squeezed states of the field for given $\bar{a},\bar{n}$. They are the only states lying on the surface of the hyperboloid H, in spite of the ambiguity in all other points of the space in Figs. 1(a) and 1(b). Such a description is not meaningless and the symmetry of the space is very important. In the case of vacuum fields ($\bar{a}=0$) the space of second-order moments is rotationally symmetric along the z axis. When $\bar{a}\neq 0$, the symmetry is broken, since the direction of the complex amplitude $\bar{a}$ is preferred, and only the mirror symmetry with respect to the plane given by the z axis and by the direction of $(\bar{a})^2$ (OV line in Fig. 1) is preserved. We also remind the reader of a more common way how to describe the noise properties of the light in the phase-space diagram.[1,15] The field (i.e., every point allowed in Fig. 1) may be represented by the complex amplitude $\bar{a}$ and the superposed noise (error) ellipse (Fig. 2), which represents the noise measurable by the homodyne detection. The correspondence between both Figs. 1 and 2 is given by the following expressions:[6,14]

$$\lambda_{1,2}^2 = 1+2(\bar{n}-|\bar{a}|^2 \pm |\overline{a^2}-(\bar{a})^2|) \;, \tag{2.4a}$$

$$\lambda^2=\lambda_1^2\cos^2(\varphi)+\lambda_2^2\sin^2(\varphi) \;, \tag{2.4b}$$

$$\varphi=\phi/2=\tfrac{1}{2}\arg[\overline{a^2}-(\bar{a})^2]-\arg(\bar{a}) \;. \tag{2.4c}$$

The coordinates of the general point P in Fig. 1 regarding the vertex V are just the variances $\bar{n}-|\bar{a}|^2$, $\overline{a^2}-(\bar{a})^2$. The meaning of the inequalities (2.2b) and (2.3) with respect to the phase-space diagram is very simple. The former one is equivalent to the relation $\lambda_1\lambda_2 \geq 1$ and equality occurs just for the TCS's and the latter one represents all fields with $\lambda_2 < 1$ and they are squeezed.

A much more complicated task is to study the connections mutually relating second-order moments with the

FIG. 2. Geometrical interpretation of the noise in an interferometric measurement in the phase-space diagram. $\bar{a}$ is the complex amplitude of the field and $\lambda_{1,2}$ are half axes of the uncertainty ellipse.

fourth-order correlation function $g^{(2)}=\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle/(\bar{n})^2$ and $(\Delta n)^2=\overline{n^2}-(\bar{n})^2=\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle-(\bar{n})^2+\bar{n}$, respectively. The following two conditions were recently derived:[6]

$$g^{(2)} \geq \frac{|\overline{a^2}|^2}{(\bar{n})^2} \tag{2.5a}$$

and

$$(\Delta n) \geq |\bar{a}|\frac{\lambda}{\lambda_1\lambda_2} \;, \tag{2.5b}$$

where the last expression depends on the parameters of the noise (error) ellipse, as sketched in Fig. 2. Both inequalities (2.5a) and (2.5b) are consequences of the GCS inequality and they forbid some fields to be antibunched.[6] All squeezed vacuum states ($\bar{a}=0$) are an evident example as a consequence of (2.3) and (2.5a), and therefore the vacuum TCS's are bunched[7] as well. However, not only for vacuum TCS's but for all TCS's there exists a quite simple relation between squeezing and antibunching, as is seen from the following geometrical interpretation. The correlation function $g^{(2)}$ can be factorized for all Gaussian[16] states (not only pure states like TCS's) to the form

$$g^{(2)}_{\text{Gauss}}=\frac{|\overline{a^2}|^2}{(\bar{n})^2}+2\left[1-\frac{|\bar{a}|^4}{(\bar{n})^2}\right] \;, \tag{2.6}$$

and the condition $g^{(2)}_{\text{Gauss}}<1$ has a very clear representation in the space in Fig. 1. Antibunched states lie inside the sphere, with the center at origin O and with the radius $\sqrt{2}|\bar{a}|^2$. All possible antibunched Gaussian states lie in the region above the hyperboloid H, inside the sphere S, and therefore they are all squeezed (i.e., they lie outside the cone C). The TCS's are situated on the surface of hyperboloid H; their antibunching reduces to the intersection of H with sphere S for all amplitudes $\bar{a}$, contrary to the approximation of the strong field in Ref. 1.

We believe that these simple geometrical considerations justify an artificial coordinate space in Fig. 1. Nevertheless, it will be useful in the following, when it will be necessary to quantify the space of "all error ellipses." We will search for the states minimizing $\overline{n^2}$ when $\bar{n}$, $\overline{a^2}$, and $\bar{a}$ are given by a straightforward generalization of the property of TCS's following from (2.2b), which minimizes $\bar{n}$ when $\overline{a^2}$ and $\bar{a}$ are given.

III. VARIATIONAL EQUATION FOR NOISE MINIMUM STATES

The previously mentioned result shows the existence of the low boundary of the correlation function $g^{(2)}$ and $(\Delta n)^2$, respectively, dependent on the first- and second-order moments $\bar{a}$, $\overline{a^2}$ and $\bar{n}$. Unfortunately, our estimations do not enable us to find the minimum value of the correlation function $g^{(2)}_{\min}(\bar{a},\overline{a^2},\bar{n})$ of the field with the given moments $\bar{a}$, $\overline{a^2}$, $\bar{n}$, and it is necessary to use the variational approach.[17,18] All states minimizing the functional $\mathrm{Tr}(\hat{\rho}\hat{n}^2)$ under the restrictions

$$\mathrm{Tr}(\hat{\rho})=1 \,, \quad \mathrm{Tr}(\hat{\rho}\hat{a})=\bar{a} \,, \quad \mathrm{Tr}(\hat{\rho}\hat{a}^2)=\overline{a^2} \,, \quad \mathrm{Tr}(\hat{\rho}\hat{n})=\bar{n} \tag{3.1}$$

are NMS's and obey the operator (Lagrange-Euler) equation

$$(\hat{n}^2+\gamma\hat{n}-\xi^2\hat{a}^{\dagger 2}-\xi^{*2}\hat{a}^2+\alpha\hat{a}^{\dagger}+\alpha^*\hat{a})\hat{\rho}_{\min}=\lambda\hat{\rho}_{\min} \,, \tag{3.2}$$

λ and γ are real parameters, and ξ, and α complex. Equation (3.2) is nothing other than the steady-state solution for the nonlinear Hamiltonian, including Kerr nonlinearity together with the generation of TCS's in the rotational-wave approximation. From the methodical point of view it is necessary, at first, to find the eigenvalue and all eigenfunctions of the ground state of such a Hamiltonian, and if some degeneration of the solution appears, then to construct an appropriate density matrix. Our strategy is straightforward—if we know $\hat{\rho}_{\min}$ and $\lambda_{\min}$ as functions of the parameters γ, α, and ξ, and if we changed them by the parameters $\bar{a}, \overline{a^2}$, and $\bar{n}$ using the relations (3.1), we could obtain $\overline{n^2}$ as a function of the parameters $\bar{a}$, $\overline{a^2}$, and $\bar{n}$ only. This provides the smallest possible value of $g^{(2)}$ and $\overline{n^2}$, respectively, when some given complex amplitudes of the field and noise are presented. Knowing this we can also predict what kinds of fields can or cannot be antibunched, and therefore the following program is to be realized: to find this function for all points above the hyperboloid H in Fig. 1. The symmetry of the space enables us to simplify the solution. In the case where $\bar{a}=0$, the function $g^{(2)}_{\min}$ will be rotationally symmetric along the z axis. When $\bar{a}\neq 0$, then mirror symmetry with respect to the plane given by the z axis and direction $(\bar{a})^2$ takes place. We will minimize the appropriately chosen nonlinear functional

$$\overline{n^2}+\gamma\bar{n}+\nu|\bar{a}|^2+\vartheta|\overline{a^2}|^2+\zeta[(\bar{a}^{\dagger})^2\overline{a^2}+(\bar{a})^2\overline{a^{\dagger 2}}] \,. \tag{3.3}$$

The above-mentioned symmetry is included and γ, ν, ϑ, and ζ are four real parameters instead of five real ones. The extremal equation[18] is given as

$$(\hat{n}^2+\gamma\hat{n}+\{[\vartheta\overline{a^2}+\zeta(\bar{a})^2]\hat{a}^{\dagger 2}+(\nu\bar{a}+2\zeta\bar{a}^{\dagger}\overline{a^2})\hat{a}^{\dagger}+\text{c.c.}\})|\psi\rangle=\lambda|\psi\rangle \,. \tag{3.4}$$

When we compare it with Eq. (3.2) we can obtain the parameters as

$$\xi^2=-[\vartheta\overline{a^2}+\zeta(\bar{a})^2] \tag{3.5a}$$

and

$$\alpha=\nu\bar{a}+2\zeta\bar{a}^{\dagger}\overline{a^2} \,. \tag{3.5b}$$

The solution has fundamental importance in quantum optics but it will be rather complicated to find it because the above-mentioned Hamiltonian includes a variety of quantum optical effects such as antibunching, squeezing, bistability, etc.

We will deal with the equation for the ground state of the wave function $|\psi\rangle$ of Eq. (3.2) and we can use advantageously the representation of the quantum mechanics in terms of functions analytic in the complex half-plane.[19] Then the creation operator may be represented by $\xi^* z$, whereas the annihilation operator by $d/d(\xi^* z)$ for $\xi\neq 0$ and the functions $(\xi^* z)^n/(n!)^{1/2}$ are then associated with the Fock state $|n\rangle$. We will search for the ground state of the equation that is regular in the complex half-plane,

$$(z^2-1)\frac{d^2 f}{dz^2}+[(\gamma+1)z+\mu^*]\frac{df}{dz}+(-\lambda+\mu|\xi|^2 z+|\xi|^4 z^2)f=0 \,, \tag{3.6}$$

where $\mu=\alpha/\xi$ is a complex parameter. This type of equation has been known since 1928 as a generalized spheroidal equation,[20] and some special cases of it came from the problem of an electron in the Coulombian potential of two nuclei, from factorization of the wave equation in oblate (prolate) spheroidal coordinates,[21] and the solution is strongly connected with problems of optimization and apodization in classical optics[22] and squeezing in quantum optics.[23] In spite of the fact that such functions are widely used, they have not yet been quite understood and new methods for studying them need to be found.[24] Nevertheless, some aspects are known. Such an equation involves three-term recursion formulas with decompositions using confluent hypergeometric functions.[25] The three-term character is also seen from the rewritten equation (3.2) in the form

$$\{(\hat{A}^{\dagger}\hat{A})^2+2\xi^*\hat{A}^{\dagger}\hat{A}^2+2\xi\hat{A}^{\dagger 2}\hat{A}+(4|\xi|^2+\gamma)\hat{A}^{\dagger}\hat{A}+[\alpha+(\gamma+1)\xi]\hat{A}^{\dagger}+[\alpha^*+(\gamma+1)\xi^*]\hat{A}-\Omega\}|\psi\rangle=0 \,, \tag{3.7}$$

$$\hat{A}=\hat{a}-\xi, \quad \Omega=\lambda+|\xi|^4-(\gamma+1)|\xi|^2-\alpha\xi^*-\alpha^*\xi \,,$$

if the basis for decomposition $|\psi\rangle$ is taken to be $\|n\rangle\rangle=(1/n!)^{1/2}(\hat{a}^{\dagger}-\xi^*)^n|\xi\rangle$, $|\xi\rangle$ is the coherent state. Alternatively, we can again represent the boson operators $\hat{A}, \hat{A}^{\dagger}$ in the complex half-plane to obtain the equation

$$z(z-2)\frac{d^2F}{dz^2}+[2|\xi|^2 z^2+(4|\xi|^2+\gamma+1)z+1+\mu^*+\gamma]\frac{dF}{dz}+[|\xi|^2(\mu+\gamma+1)z-\Omega]F=0 \tag{3.8}$$

and for the decomposition it holds that $F(z)=\sum_{n=0}^{\infty}a_n z^n$, $a_{-1}=0$, and

$$[n(n-1)+n(4|\xi|^2+\gamma+1)-\Omega]a_n+(n+1)[2n+\gamma+1+\mu^*]a_{n+1}+|\xi|^2(2n-1+\mu+\gamma)a_{n-1}=0 \,. \tag{3.9}$$

If we adopt a general technique for solving such equations,[21] the normalizable solution exists for a discrete set of parameters Ω only, and appropriate values are obtained from the condition for a continued fraction

$$X_n=\frac{a_n}{a_{n-1}},\quad n>0,\quad X_1=\frac{\Omega}{1+\gamma+\mu^*},$$
$$n(n-1)+n(4|\xi|^2+\gamma+1)-\Omega+(n+1)(2n+1+\gamma+\mu^*)X_{n+1}+(1/X_n)|\xi|^2(2n-1+\gamma+\mu)=0\,. \quad (3.10)$$

The solution belongs to higher transcendental functions and includes, as for the special cases, prolate spheroidal functions ($\alpha=0$), Mathieu functions ($\alpha=0$, $\gamma=0$), and polynomial solutions ($\mu+\gamma=-2N+1$; $N=1,2,\ldots$). We can solve the equation for some values of parameters explicitly and obtain, in such a way, information about the behavior for the function $g^{(2)}_{\min}$.

IV. SPECIAL SOLUTIONS

The very important set of special solutions is represented by all TCS's, as it follows from the unambiguity in (2.2b) under the restriction

$$|\overline{a^2}-(\bar a)^2|^2=(\bar n-|\bar a|^2)^2+\bar n-|\bar a|^2 \quad (4.1)$$

for all $\bar a$ (i.e., the surface of hyperboloid H in Fig. 1), or alternatively, from Eq. (3.2) for $\gamma\to\infty$, but with $|\xi|^2/\gamma$, $\alpha/\gamma\neq0$. The value of $g^{(2)}_{\min}$ is expressed as (2.6) under the condition (4.1).

Another well-known example of the generalization of the Fock states may be given. The Fock state is the eigenfunction of (3.2) for $\alpha=\xi=0$, and the solution can be twice degenerated for appropriate values of γ. The minimum state appears for $\gamma=-2[\bar n]-1$ and for the density matrix

$$\hat\rho_{\min}=d|[\bar n]\rangle\langle[\bar n]|+(1-d)|[\bar n]+1\rangle\langle[\bar n]+1|+(\{\bar a/([\bar n]+1)^{1/2}\}|[\bar n]+1\rangle\langle[\bar n]|+\text{c.c.})\,, \quad (4.2)$$

where $d=\bar n-[\bar n]$ and $[\bar n]$ is the integer part of the mean photon number $\bar n$. Such states provide the minimum value

$$(\Delta n)^2_{\min}=d(1-d)\le\tfrac14\,, \quad (4.3)$$

depending on the mean photon number only. It holds for all complex amplitudes satisfying (1.1b), i.e.,

$$|\bar a|^2\le([\bar n]+1)d(1-d)\,. \quad (4.4)$$

In Fig. 1 such minimum states are situated (i) for $\bar a=0$ along the z axis without constraints, and (ii) for $\bar a\neq0$ along the z axis inside the closed intervals

$$\langle[\bar n]-\tfrac12-[\tfrac14-|\bar a|^2/([\bar n]+1)]^{1/2}\,;[\bar n]-\tfrac12+[\tfrac14-|\bar a|^2/([\bar n]+1)]^{1/2}\rangle\quad\text{for } [\bar n]\ge4|\bar a|^2-1\,.$$

The relation (4.3) was previously derived in Ref. 26 using the heuristic method based on the inequality

$$\mathrm{Tr}[\hat\rho(\hat n-k)(\hat n-k-1)]\ge0\quad\text{for all } k=0,1,2,\ldots\,. \quad (4.5)$$

The third example can be treated as the minimizing of $\langle\hat a^{\dagger2}\hat a^2\rangle$ for the given $\overline{a^2}$ only. Variational formulations need $\alpha=0$, $\gamma=-1$, and we search for the ground state of the equation

$$(\hat a^{\dagger2}-\xi^{*2})(\hat a^2-\xi^2)|\psi\rangle=(\lambda-|\xi|^4)\,. \quad (4.6)$$

It is evident that coherent states $|+\xi\rangle,|-\xi\rangle$ create a subspace of extremal states and they also directly fulfill the equality sign in the inequality (2.5a). Advantageously we will define the new orthonormal basis as eigenfunctions

$$|e\rangle=\frac{\exp(|\xi|^2/2)}{2(\cosh|\xi|^2)^{1/2}}(|+\xi\rangle+|-\xi\rangle)=(\cosh|\xi|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{\xi^{2n}}{(2n!)^{1/2}}|2n\rangle \quad (4.7a)$$

and

$$|o\rangle=\frac{\exp(|\xi|^2/2)}{2(\sinh|\xi|^2)^{1/2}}(|+\xi\rangle-|-\xi\rangle)=(\sinh|\xi|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{\xi^{2n+1}}{[(2n+1)!]^{1/2}}|2n+1\rangle\,, \quad (4.7b)$$

including even (odd) Fock states only. They are also known as SU(1,1) coherent states[27] or even (odd) coherent states.[16,28] The annihilation operator acts on the subspace according to the rule

$$\hat a|e\rangle=\xi(\tanh|\xi|^2)^{1/2}|o\rangle\,, \quad (4.8a)$$
$$\hat a|o\rangle=\xi(\tanh|\xi|^2)^{-1/2}|e\rangle\,. \quad (4.8b)$$

The extremal density matrix is then given as

$$\hat\rho_{\min}=p|e\rangle\langle e|+(1-p)|o\rangle\langle o|+r|e\rangle\langle o|+r^*|o\rangle\langle e|\,, \quad (4.9)$$

and the parameters ξ, p, and r are connected with $\bar a$, $\overline{a^2}$, $\bar n$, and $\langle\hat a^{\dagger2}\hat a^2\rangle$ through the relations

$$\begin{aligned}&\bar a=\xi\{[r(\tanh|\xi|^2)^{1/2}+r^*(\tanh|\xi|^2)^{-1/2}]\}\,,\\&\overline{a^2}=\xi^2\,,\\&\bar n=|\xi|^2[p\tanh|\xi|^2+(1-p)/\tanh|\xi|^2]\,,\\&\langle\hat a^{\dagger2}\hat a^2\rangle=|\xi|^4\,,\end{aligned} \quad (4.10)$$

and this description is correct for

$$0\le p\le1\,, \quad (4.11a)$$
$$|r|^2\le p(1-p)\,, \quad (4.11b)$$

as the consequences of the properties (1.1a) and (1.1b). The dependence in (4.10) can be inverted as

$$p=\frac{\sinh 2|\overline{a^2}|}{2|\overline{a^2}|}\left[\frac{|\overline{a^2}|}{\tanh|\overline{a^2}|}-\bar{n}\right], \quad (4.12a)$$

$$|r|^2=\frac{|\bar{a}|^2\sinh 4|\overline{a^2}|}{4|\overline{a^2}|}[1-\cos(\chi)\tanh 2|\overline{a^2}|], \quad (4.12b)$$

where $\chi=\arg(\overline{a^2})-2\arg(\bar{a})$. Consequently, the conditions (4.11), and (4.12) mark out the region in the space of second-order moments, where $\hat{\rho}_{\min}$ defines the minimal value

$$g^{(2)}_{\min}=\frac{|\overline{a^2}|^2}{(\bar{n})^2}. \quad (4.13)$$

The region is defined as

$$\left[\frac{1}{\tanh|\overline{a^2}|}-\frac{\bar{n}}{|\overline{a^2}|}\right]\left[-\tanh|\overline{a^2}|+\frac{\bar{n}}{|\overline{a^2}|}\right] \geq \frac{1-\cos(\chi)\tanh 2|\overline{a^2}|}{\tanh 2|\overline{a^2}|}2|\bar{a}|^2/|\overline{a^2}|. \quad (4.14a)$$

All such points lie near the zero cone C_0 but above the hyperboloid H. Effectively the largest region occurs for $\bar{a}=0$,

$$|\overline{a^2}|\tanh|\overline{a^2}|\leq\bar{n}\leq|\overline{a^2}|(\tanh|\overline{a^2}|)^{-1}, \quad (4.14b)$$

and we can summarize our knowledge about $g^{(2)}_{\min}$ in Fig. 3 in this case by the following.

(i) $\overline{a^2}=0$; $g^{(2)}_{\min}=1-1/\bar{n}+d(1-d)/(\bar{n})^2$; $\hat{\rho}_{\min}$ is given in (4.2).

(ii) $0\leq\bar{n}\leq|\overline{a^2}|\tanh|\overline{a^2}|$; $g^{(2)}_{\min}<1$; $\hat{\rho}_{\min}$ includes polynomial solutions of (3.2) for $\gamma<-1$, explicitly unknown.

(iii) $|\overline{a^2}|\tanh|\overline{a^2}|\leq\bar{n}\leq|\overline{a^2}|(\tanh|\overline{a^2}|)^{-1}$; $g^{(2)}_{\min}=|\overline{a^2}|^2/(\bar{n})^2$; $\hat{\rho}_{\min}$ is given in (4.9).

(iv) $\bar{n}\leq|\overline{a^2}|\tanh|\overline{a^2}|)^{-1}$; $g^{(2)}_{\min}>1$; $\hat{\rho}_{\min}$ includes the solution for spheroidal wave functions for $\gamma>-1$, explicitly unknown.

(v) $\bar{n}(\bar{n}+1)=|\overline{a^2}|^2$; $g^{(2)}_{\min}=3+1/\bar{n}$; $\hat{\rho}_{\min}$ is vacuum TCS.

In cases (ii) and (iv) it is necessary to solve the spheroidal equations (3.2) and (3.6), respectively, for α, $\mu=0$, as a consequence of the symmetry (3.5) for $\bar{a}=0$.

From the point of view of quantum optics there exists an important surface $Q_{\bar{a}}$ defined as

$$g^{(2)}_{\min}(\bar{a},\overline{a^2},\bar{n})=1, \quad (4.15)$$

which separates the region of possible antibunched states ($g^{(2)}_{\min}<1$) from the region where all states must be bunched ($g^{(2)}_{\min}>1$). Such a surface is zero cone C_0 in the case $\bar{a}=0$, and when $|\bar{a}|$ increases, the surface is deformed and only the front part of C_0 with respect to the complex amplitude $(\bar{a})^2$ is preserved, according to the condition (4.14a) for $|\overline{a^2}|=\bar{n}$,

$$1-\tanh|\overline{a^2}|\geq\frac{|\bar{a}|^2}{|\overline{a^2}|}[1-\cos(\chi)\tanh 2|\overline{a^2}|]. \quad (4.16)$$

The structure of the surface $Q_{\bar{a}}$ for all intermediate states ($\bar{a}\neq 0,\infty$) will probably be complicated, but there are some reasons for its simple determination for strong fields because condition (4.16) reduces to half a straight line $\chi=0$, $|\overline{a^2}|\geq|\bar{a}|^2$, in this case. We will try to find Q_∞ in the future.

As the last example we will deal with the amplitude squeezed states[9] decreasing the variation $(\Delta n)^2$ up to $(\bar{n})^{1/3}$. Such states are generated in a nonlinear Mach-Zehnder interferometer using the Kerr medium and they may be expressed as

$$|\psi\rangle=\hat{D}(\eta)\hat{U}(\omega)|\alpha_1\rangle, \quad (4.17)$$

where the displacement operator $\hat{D}(\eta)$ and the operator

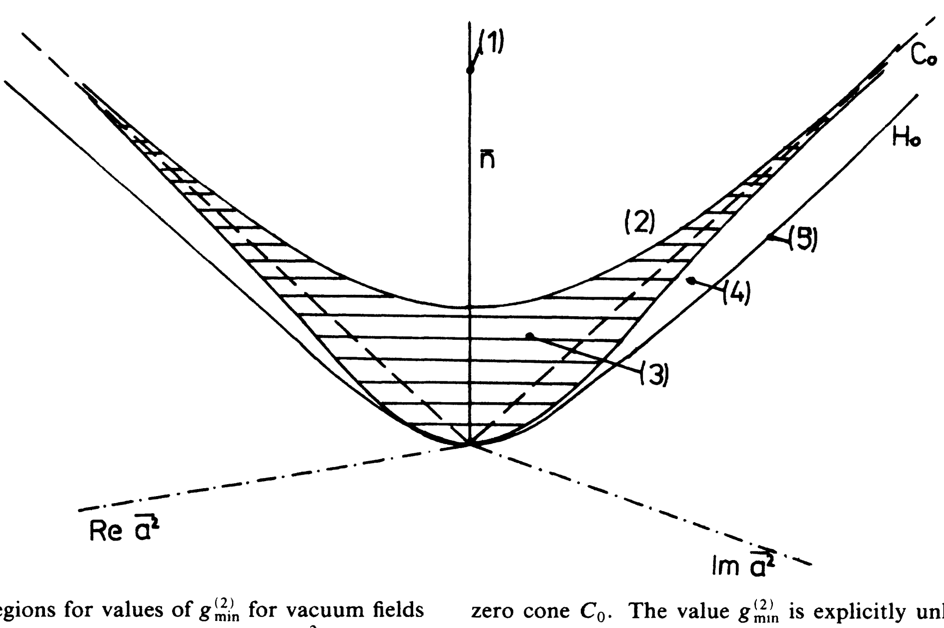

FIG. 3. The five regions for values of $g^{(2)}_{\min}$ for vacuum fields ($\bar{a}=0$). Inside C_0 the field can be antibunched ($g^2_{\min}<1$) but outside it cannot ($g^{(2)}_{\min}>1$). The surface Q_0 is incident with the zero cone C_0. The value $g^{(2)}_{\min}$ is explicitly unknown inside the regions (2) and (4).

$\hat{U}(\omega)=\exp[\frac{1}{2}i\omega\hat{n}(\hat{n}-1)]$ act on the coherent state $|\alpha_1\rangle$. The parameters are suitably chosen to fulfill the conditions

$$\eta \simeq -i\alpha_1 e^{i\omega n_c}/n_c^{1/3}, \quad \omega \simeq n_c^{-2/3}, \quad n_c=|\alpha_1|^2 , \tag{4.18a}$$

so that $n_c\omega \gg 1$, $n_c\omega^2 \ll 1$ takes place for n_c large. Under these conditions the lower-order moments can be approximated as

$$\begin{aligned} &|\bar{a}| \simeq n_c^{1/2}, \quad \langle \Delta\hat{a}^\dagger \Delta\hat{a}\rangle \simeq |\langle(\Delta\hat{a})^2\rangle| \simeq n_c^{2/3} ,\\ &\langle(\Delta\hat{a})^2\rangle/(\bar{a})^2 \simeq -n_c^{-1/3} . \end{aligned} \tag{4.18b}$$

This state is not evidently the extremal state for (3.2) in the mathematical sense, but it is reasonable to investigate how closely it approximates the state with a minimum rms of $\hat{n}$ for given large moments $\bar{a}$ and $\bar{n}$. This suggestion seems to be in accordance with the requirement of symmetry (3.5a), since for $\xi=0$ the condition for $\overline{a^2}/(\bar{a})^2$ to be real applies. We will compare the state (4.17) with the solution of Eqs. (3.2) and (3.6), respectively, for $\xi=0$, including the symmetry of Eq. (3.3),

$$[\hat{n}^2+\gamma\hat{n}+v(\bar{a}^\dagger\hat{a}+\bar{a}\hat{a}^\dagger)]|\Psi\rangle=\lambda|\Psi\rangle . \tag{4.19}$$

Unfortunately we do not know the solution explicitly and therefore our estimation will be based on the higher-order GCS inequalities (2.1), for which the equality sign occurs just for the NMS's. The method is given in the Appendix, and we can conclude that for n_c large (4.17) seems to be quite a good approximation of the solution.

V. REMARKS ON THE SOLUTION

In this section we will mention the following as yet undiscussed possibilities of other studies concerning NMS's.

(i) We can deduce from the variational nature of Eq. (3.2) that a similar equation will also solve the problem of finding the maximum squeezing under given conditions. Therefore relevant processes connected with the Hamiltonian (3.2) can, in special cases, improve squeezing, as is investigated elsewhere.[29].

(ii) The inequality (2.5b) is valid, but the extremal states have not yet been found,[30] besides the simplest case of Fock states. It is interesting to note that the equality occurs for TCS's under the approximation of a strong field used in Ref. 1, Chap. 3.3 for $\bar{n} \gg \exp(2s)$, where s is parameter of squeezing $\sinh^2 s=\langle \Delta\hat{a}^\dagger\Delta\hat{a}\rangle$.

(iii) It is possible that the ground steady-state solution of (3.2) may not provide the minimum value of the rms of $\hat{n}$ for all possible points above the hyperboloid H. Then it is necessary to find another solution of (3.2) with the next value of the parameter $\lambda_1 > \lambda_{\min}$, and to do the previous procedure until the minimum value of the rms of $\hat{n}$ on the whole configuration space in Fig. 1 is found. The solution for NMS's is really the ground state of (3.2) for points on the surface of the hyperboloid H (TCS), but on the contrary we must use the entire spectrum in the case of Fock states situated on the z axis. It is still unknown if some discontinuities of $(\Delta n)^2$ can appear, as a consequence of different steady-state solutions of the Hamiltonian (3.2).

VI. CONCLUSIONS

This paper is related to the fundamental problem of quantum mechanics and can be characterized by a basic question: What is the connection between mathematical tools of quantum mechanics and really measurable quantities? We solve it in the framework of quantum optics, using the connection between the properties of the density matrix and moments of the field operators up to the fourth order. NMS's introduced in this paper represent extremal states with respect to both nonclassical effects squeezing and antibunching. Unlike number-phase MUS's they are tightly connected to experimental conditions, since noise properties in homodyne detection are better understood than quantum phase properties. NMS's allows us to clarify the question of maximum possible antibunching.

(a) The value $(\bar{n})^{2/3}$ represents the minimum value $(\Delta n)^2$ under the constraints that $\bar{a}$ is given and condition (4.1) applies.

(b) The value $(\bar{n})^{1/3}$ represents an approximation of the value $(\Delta n)^2$ under the constraints that $\bar{a}$ and $\bar{n}$ are given and condition (4.4) is not true.

In both cases the limit of a strong field with $\bar{a}$ large is supposed. To summarize, we can find the following important reasons for studying NMS's in spite of their mathematically complicated development.

(i) They entirely replace the concept of number-phase MUS's. They are realizable, the problems concerning phase are avoided, and information about the rms of $\hat{n}$ is preserved.

(ii) Both effects, squeezing and antibunching, have been treated up to this time as independent phenomenon of different orders. It would be inconsistent if we did not clarify the intrinsic restrictions following from quantum mechanics. NMS's are able to provide information on the extent to which squeezing and antibunching are compatible. Particularly, for vacuum fields ($\bar{a}=0$), they are complementary effects.

(iii) The concept of NMS's unifies many states of light in the common framework.

Besides the fundamental questions of quantum optics more practical problems have arisen as well. For instance, both quantum effects could be simultaneously utilized in quantum nondemolition measurement and in optical communication systems in the future. We also suppose that the above considerations provide new traits for a better understanding of the nature of the quantum noise of light.

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APPENDIX

The following inequality takes place and the equality sign holds just for extremal states (4.19):

$$\frac{|\langle(\Delta\hat{n}^2)(\Delta\hat{n}+\frac{\nu}{\gamma}\bar{a}\Delta\hat{a}^\dagger+\frac{\nu}{\gamma}\bar{a}^\dagger\Delta\hat{a})\rangle|^2}{\langle(\Delta\hat{n}^2)^2\rangle\langle(\Delta\hat{n}+\frac{\nu}{\gamma}\bar{a}\Delta\hat{a}^\dagger+\frac{\nu}{\gamma}\bar{a}^\dagger\Delta\hat{a})^2\rangle}\le 1\ , \quad \text{(A1)}$$

where $\Delta\hat{A}=\hat{A}-\bar{A}$. The expression on the left-hand side of (A1) is the measure of closeness to the extremal state of the given state of the field. In this spirit we could also derive seven inequalities, if we were interested in extremal states of (3.2). But we will deal with the (A1) inequality only and restrict ourselves to the highest order in the complex field amplitude $|\alpha_1|$. The fraction on the left-hand side can be easily developed as

$$\frac{|\alpha_1|^2(2|\alpha_1|^2+1)^2}{|\alpha_1|^2[(2|\alpha_1|^2+1)^2+2|\alpha_1|^2]}\simeq\left[1+\frac{1}{2|\alpha_1|^2}\right]^{-1}, \quad \text{(A2)}$$

which seems to be quite a good approximation of 1 in the case of a strong field.

It is interesting to note that Eq. (4.19) enables us to factorize some moments in the closed form, if we act on both sides by the operators $\Delta\hat{a},\Delta\hat{a}^\dagger,\Delta\hat{n},\Delta\hat{n}^2$ and then take the mean value. The following factorization takes place:

$$\begin{aligned}\langle\Delta\hat{a}^\dagger\Delta\hat{n}\rangle&=-\tfrac{1}{2}\bar{a}^\dagger(2\bar{n}+v+\gamma+1)\ ,\\ \langle\Delta\hat{a}^\dagger\Delta\hat{n}^2\rangle&=-\nu(\bar{a}^{\dagger 2}\bar{a}+\bar{a}^\dagger\bar{n}-2|\bar{a}|^2\bar{a}^\dagger)-\gamma\langle\Delta\hat{a}^\dagger\Delta\hat{n}\rangle\ ,\\ \langle\Delta\hat{n}^2\Delta\hat{n}\rangle&=v|\bar{a}|^2(2\bar{n}+v)-\gamma[\overline{n^2}-(\bar{n})^2]\ ,\\ \langle(\Delta\hat{n}^2)^2\rangle&=-\gamma\langle\Delta\hat{n}^2\Delta\hat{n}\rangle-\nu(\bar{a}\langle\Delta\hat{a}^\dagger\Delta\hat{n}^2\rangle+\text{c.c.})\\ &\quad-2\nu\bar{a}\langle\Delta\hat{a}^\dagger\Delta\hat{n}\rangle-\nu|\bar{a}|^2(2\bar{n}+1)\ .\end{aligned} \quad \text{(A3)}$$

These four relations are entirely equivalent to the solution of Eq. (4.19), because such moments provide the equality in (A1).

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