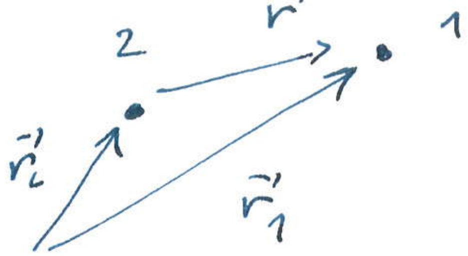
2 tělesa



$$\vec{a}_1 = \frac{\vec{F}_{12}}{m_1}$$ – síla 2 na 1

$$\vec{a}_2 = -\frac{\vec{F}_{12}}{m_2}$$

$$\vec{a} = \vec{a}_1 - \vec{a}_2 = \vec{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \vec{F}_{12}\,\frac{m_1 + m_2}{m_1 m_2}$$

$$m\vec{a} = \vec{F}_{12}, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$ (redukovaná hmotn.)

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grav.

grav. parametr

$$m\vec{a} = -\frac{G m_1 m_2}{r^3}\vec{r} = -\frac{G(m_1+m_2)m}{r^3}\vec{r} = -\frac{\mu m}{r^3}\vec{r}$$

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kinetická energie

$$E_k = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$ (soustava spoj. s těžištěm)

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0 \qquad m_1(\vec{r} + \vec{r}_2) + m_2 \vec{r}_2 = 0$$

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \qquad \vec{r}_2 = -\frac{m_1 \vec{r}}{m_1 + m_2}$$

$$E_k = \frac{1}{2} m_1 \left(\frac{m_2 \dot{r}}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{-m_1 \dot{r}}{m_1 + m_2} \right)^2$$

$$= \frac{1}{2} \dot{r}^2 \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2}$$

$$= \frac{1}{2} \dot{r}^2 \frac{m_1 m_2}{m_1 + m_2}$$

$$= \frac{1}{2} m \dot{r}^2$$

pot. energ.

$$E_p = -\frac{G m_1 m_2}{r} = -\frac{\mu m}{r}$$

$$\mu = G(m_1 + m_2)$$

dráha v rovině

$\vec{k} = \vec{r} \times \dot{\vec{r}}$ (moment rychlosti)

$$\dot{\vec{k}} = \vec{r} \times \ddot{\vec{r}} + \underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{0}$$

$$= -\frac{\mu}{r^3} \vec{r} \times \vec{r} = 0$$

$\vec{k} =$ konst. vektor $\perp$ $\vec{r}$

$\Rightarrow$ pohyb v rovině

energie dráhy (specifická)

platí $\underline{\underline{\dot{\vec{r}}\cdot\vec{r}}} = \frac{1}{2}(\vec{r}\cdot\vec{r})^{\cdot} = \frac{1}{2}(r^2)^{\cdot} = \underline{\underline{r\dot{r}}}$

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$$\dot{\vec{r}}\cdot\ddot{\vec{r}} = -\mu\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3} = -\mu\frac{\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{\mu}{r}\right)$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}} = \frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0$$

$$\Downarrow \quad \underline{\underline{\frac{1}{2}v^2 - \frac{\mu}{r}}} = \text{konst.}$$

1. Kepl. zákon altern.

$$\vec{k} = \vec{r}\times\dot{\vec{r}} = \text{konst.}$$

$$\vec{k}\times\ddot{\vec{r}} = (\vec{r}\times\dot{\vec{r}})\times\left(-\frac{\mu}{r^3}\vec{r}\right)$$

$$= -\frac{\mu}{r^3}\Big[\dot{\vec{r}}\underbrace{(\vec{r}\cdot\vec{r})}_{r^2} - \vec{r}\underbrace{(\vec{r}\cdot\dot{\vec{r}})}_{r\dot{r}}\Big]$$

$$= -\mu\left[\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\,\dot{r}}{r^2}\right] = \frac{d}{dt}\left(-\mu\frac{\vec{r}}{r}\right)$$

dále

$$\vec{k}\times\ddot{\vec{r}} = \frac{d}{dt}\left(\vec{k}\times\dot{\vec{r}}\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\vec{k}\times\dot{\vec{r}} + \mu\frac{\vec{r}}{r}\right) = 0$$

$$\vec{k}\times\dot{\vec{r}} + \mu\frac{\vec{r}}{r} = -\mu\vec{e} = \text{konst}$$

$$\vec{r}\cdot\vec{e} = r\,e\cos\varphi$$

$$\vec{r}\cdot\vec{e} = -\frac{1}{\mu}\left[\vec{r}\cdot(\vec{k}\times\dot{\vec{r}}) + \mu\frac{\vec{r}\cdot\vec{r}}{r}\right]$$

$$= -\frac{1}{\mu}\left[\vec{k}\cdot(\underbrace{\dot{\vec{r}}\times\vec{r}}_{-\vec{k}}) + \mu r\right]$$

$$= \frac{k^2}{\mu} - r$$

$$\Downarrow$$

$$r e \cos\varphi = \frac{k^2}{\mu} - r$$

$$r = \frac{k^2/\mu}{1 + e\cos\varphi}$$

bylo použito

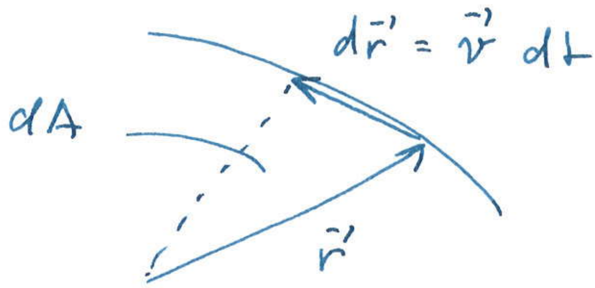
$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{a}(\vec{b}\cdot\vec{c})$$

$\vec{a}\cdot(\vec{b}\times\vec{c})$ invar. vzhledem k rotaci $\vec{a},\vec{b},\vec{c}$

II Keplerův zákon

spec. moment hybnosti

$$\vec{k} = \vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{v} = \text{konst.}$$



$$dA = \frac{|\vec{r} \times \vec{v}|\, dt}{2}$$

$\Downarrow$

$$\boxed{\frac{dA}{dt} = \frac{k}{2} = \text{konst.}}$$

energie orbity – odvoz. (jinak

$$\frac{1}{2}v^2 - \frac{\mu}{r} = \text{const.} = \frac{1}{2}v_{max}^2 - \frac{\mu}{r_{min}} \quad \Leftrightarrow$$

$$r = \frac{a(1-e^2)}{1+e\cos\nu} = \frac{h^2/\mu}{1+e\cos\nu}$$

$$r_{min} = a(1-e)$$

$$\frac{h^2}{\mu} = a(1-e^2) \Rightarrow h^2 = \mu a(1-e^2)$$

$$h^2 = r_{min}^2 v_{max}^2 \Rightarrow v_{max}^2 = \frac{h^2}{r_{min}^2}$$

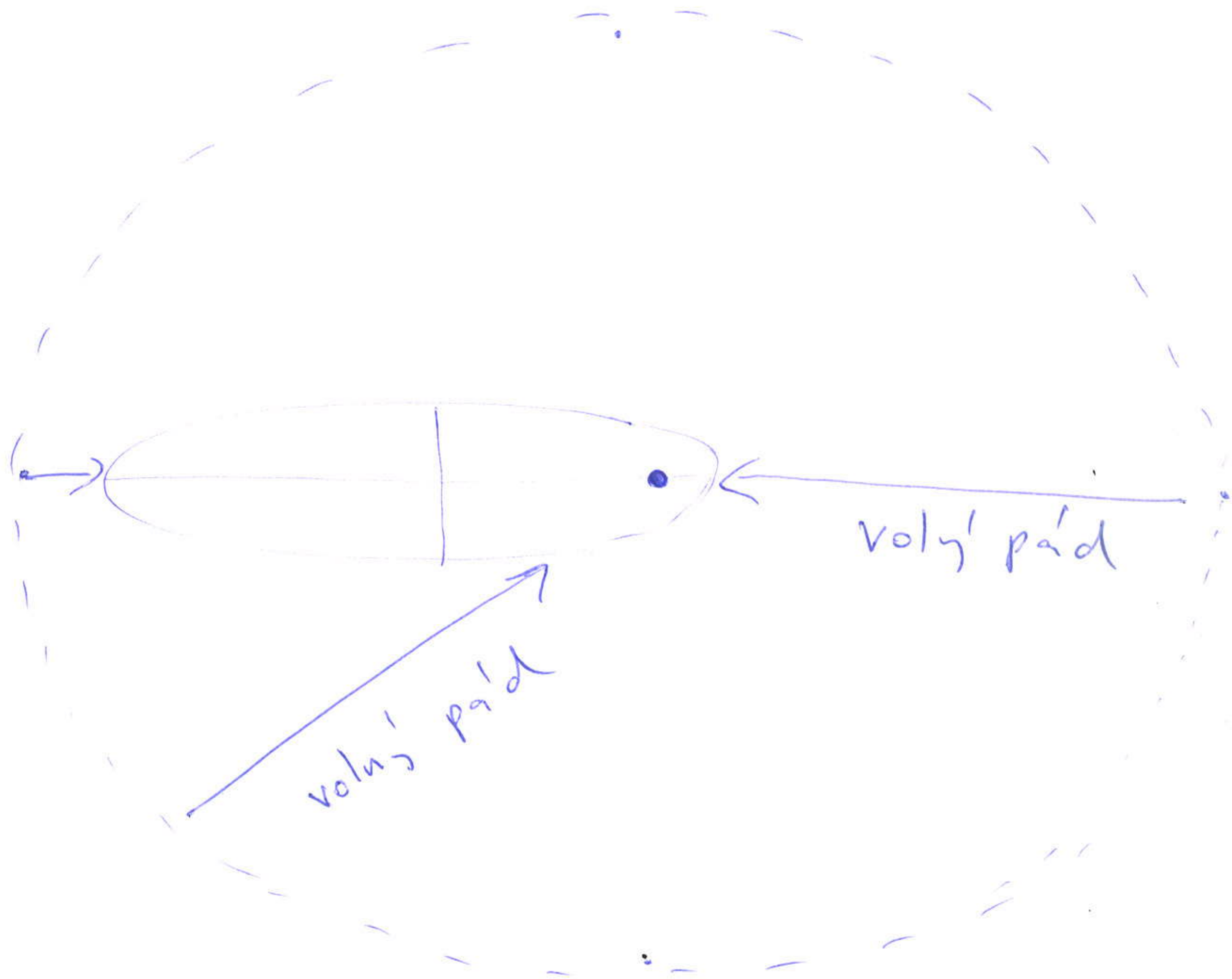
$$\Leftrightarrow \quad \frac{1}{2}\frac{\mu \cancel{a}(1-e^2)}{a^{\cancel{2}}(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$= \underline{\underline{-\frac{\mu}{2a}}}\left[\underbrace{\frac{-(1-e^2)+2(1-e)}{(1-e)^2}}_{\frac{e^2-1+2-2e}{(1-e)^2} = 1}\right]$$

interpretace

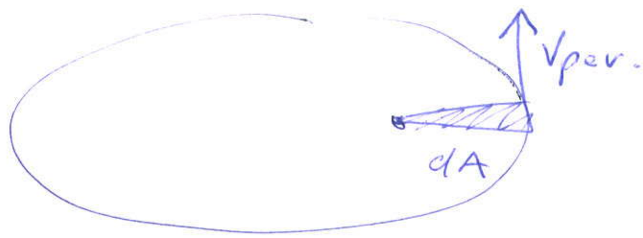
$$\frac{1}{2} v^2 = \frac{\mu}{r} - \frac{\mu}{2a}$$

$E_{kin.}$ $\Delta E_{pot.}$



odvození vztahu $p = \frac{h^2}{\mu}$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$



$$\frac{dA}{dt} \equiv \frac{h}{2}$$

Spec. případ perihelium

$$\frac{dA}{dt} = \frac{1}{2} r_{per} v_{per} = \frac{1}{2} a(1-e) \sqrt{\mu} \sqrt{\frac{2}{a(1-e)} - \frac{1}{a}}$$

$$\frac{h}{2} = \frac{\sqrt{\mu}}{2} a(1-e) \sqrt{\frac{2-(1-e)}{a(1-e)}}$$

$$h = \sqrt{\mu}\sqrt{a}\,(1-e)\sqrt{\frac{1+e}{1-e}}$$

$$h = \sqrt{\mu}\sqrt{a}\sqrt{(1+e)(1-e)}$$

$$h = \sqrt{\mu}\sqrt{a(1-e^2)}$$

$$h = \sqrt{\mu}\sqrt{p}$$

$$\boxed{p = \frac{h^2}{\mu}}$$

odvození III Keplerova zákona

$$\frac{dA}{dt} = \frac{h}{2} \quad , \quad h = \sqrt{\mu p} = \sqrt{\mu \frac{b^2}{a}}$$

za jednu otočku:

$$\pi a b = \frac{h}{2} P$$

$$P = \frac{2\pi a b}{h} = \frac{2\pi a \cancel{b}}{\sqrt{\mu \frac{\cancel{b^2}}{a}}} = \boxed{2\pi \sqrt{\frac{a^3}{\mu}}}$$

Sling shot

rychlosti

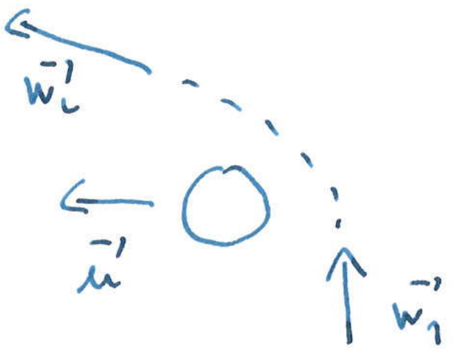
$\vec{w}_1$ ... heliocentr. před

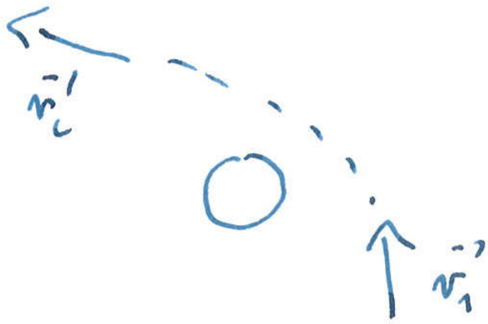
$\vec{w}_2$ ... -‖- po

$\vec{r}_1$ .. planetocentr. před

$\vec{r}_2$ ... -‖- po

$\vec{u}$ ... heliocentr. planety





helio..

planeto..

$$\vec{r}_1 = \vec{w}_1 - \vec{u} \qquad \vec{w}_1 = \vec{r}_1 + \vec{u}$$

$$\vec{r}_2 = \vec{w}_2 - \vec{u} \qquad \vec{w}_2 = \vec{r}_2 + \vec{u}$$

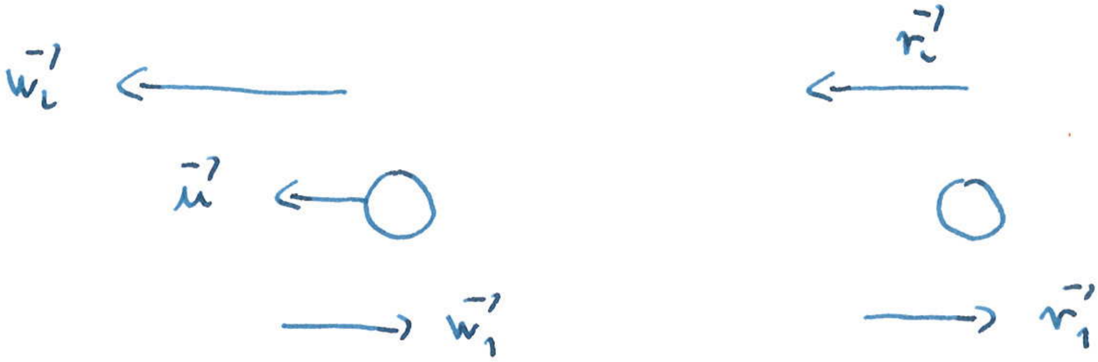
$|\vec{r}_1| = |\vec{r}_2| = r$ elast. srážka

heliocentr. soustava

před   po

$\Rightarrow \quad |\vec{w}_2| > |\vec{w}_1|$

spec. případ



$\vec{v}_2 = -\vec{v}_1$

$|\vec{w}_1| = |\vec{v}_1 + \vec{u}| = v - u.$

$|\vec{w}_2| = |\vec{v}_2 + \vec{u}| = v + u$

$\Delta w = w_2 - w_1 = 2u$