G – J eliminace

$$\left.\begin{aligned} A x &= b \\ A' x &= b' \end{aligned}\right\}$$ ekvivalentní soustavy

$$A x = b \qquad\qquad A A^{-1} = \hat{1}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\hat{1} x = x \qquad\qquad \hat{1} A^{-1} = A^{-1}$$

tj. transformuji A na $\hat{1}$ operacemi zachovávajícími řešení soustavy

$\swarrow \qquad\qquad \searrow$

řešení x $\qquad\qquad$ inverzní matice A^{-1}

LU 3x3

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$j=1$

$i=1 \qquad a_{11} = u_{11}$

$i=2 \qquad a_{21} = l_{21} u_{11}$

$i=3 \qquad a_{31} = l_{31} u_{11}$

$j=2$

$i=1 \qquad a_{12} = u_{12}$

$i=2 \qquad a_{22} = l_{21} u_{12} + u_{22}$

$a_{32} = l_{31} u_{12} + l_{32} u_{22}$

$j=3$

$i=1 \qquad a_{13} = u_{13}$

$a_{23} = l_{21} u_{13} + u_{23}$

$a_{33} = l_{31} u_{13} + l_{32} u_{23} + u_{33}$

LU – pivotace

$\det \underline{A} \neq 0$ ale $a_{11} = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \Rightarrow u_{11} = 0$$

čili $\operatorname{Det} \underline{A} = 0$

pivotace

- přehození řádků

$$LU = PA$$

permutační matice

- řešení soustavy

$$Ax = b$$

$$PAx = Pb \equiv b'$$

$$\boxed{LUx = b'}$$

před zpětným dosaz. permutuji pravou stranu!

determinant

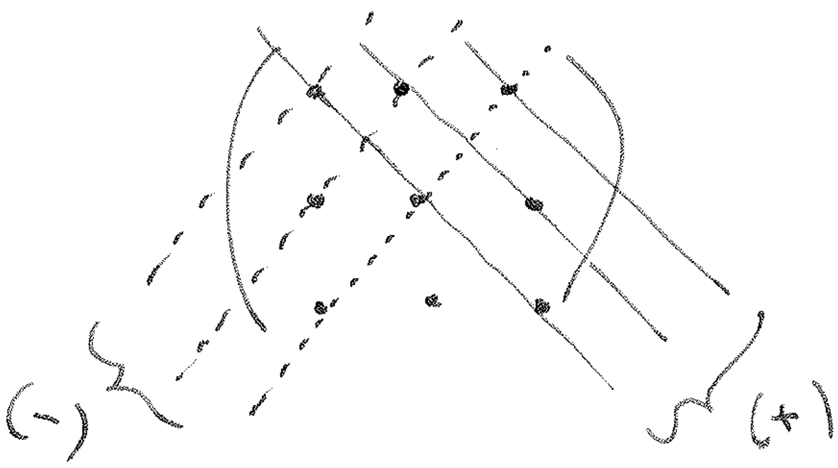
permutace indexů

např. dim 3

123	
1 2 3	(+)
1 3 2	(−)
3 2 1	(−)
2 1 3	(−)
2 3 1	(+)
3 1 2	(+)

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$$



Cramer. pravidlo

$$\underbrace{\begin{pmatrix} \begin{pmatrix} \vec{a}_1 \end{pmatrix} & \begin{pmatrix} \vec{a}_2 \end{pmatrix} & \cdots \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vec{b} \end{pmatrix}$$

$$\vec{b} = \sum_j x_j \vec{a}_j$$

$$A_i = \begin{pmatrix} \begin{pmatrix} \vec{a}_1 \end{pmatrix} & \cdots & \begin{pmatrix} \vec{a}_{i-1} \end{pmatrix} & \begin{pmatrix} \vec{b} \end{pmatrix} & \begin{pmatrix} \vec{a}_{i+1} \end{pmatrix} & \cdots \end{pmatrix}$$

$$= \sum_j \begin{pmatrix} \begin{pmatrix} \vec{a}_1 \end{pmatrix} & \cdots & \begin{pmatrix} \vec{a}_{i-1} \end{pmatrix} & \begin{pmatrix} x_j \vec{a}_j \end{pmatrix} & \begin{pmatrix} \vec{a}_{i+1} \end{pmatrix} & \cdots \end{pmatrix}$$

$$\underline{\underline{\det A_i}} = \det \begin{pmatrix} \begin{pmatrix} \vec{a}_1 \end{pmatrix} & \cdots & \begin{pmatrix} \vec{a}_{i-1} \end{pmatrix} & \begin{pmatrix} x_i \vec{a}_i \end{pmatrix} & \begin{pmatrix} \vec{a}_{i+1} \end{pmatrix} & \cdots \end{pmatrix}$$

$$= \underline{\underline{x_i \det A}}$$

interpretace SVD

$$(USV^T)_{mn} = \sum_k \sum_l U_{mk} S_{kl} V_{nl} =$$

$$S_{kl} = s_k \delta_{kl}$$

$$= \sum_k s_k U_{mk} V_{nk} = \sum_k s_k \underbrace{u^k_m v^k_n}_{(u^k \otimes v^k)_{mn}}$$

$$USV^T = \sum_k s_k \, u^k \otimes v^k$$

$$\equiv \sum_k s_k |u_k\rangle\langle v_k|$$

---

nebo

$$USV^T = \left(\sum_k |u_k\rangle\langle k|\right)\left(\sum_l s_l |l\rangle\langle l|\right)\left(\sum_m |m\rangle\langle v_m|\right)$$

výpočetní báze $\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \dots$

$$= \sum_k s_k |u_k\rangle\langle v_k| = \sum_k^r s_k |u_k\rangle\langle v_k|$$

pseudoinverze

$$A x = b \quad , \quad A = \sum_{k=1}^{r} s_k |u_k\rangle\langle v_k| \; ; \; s_k > 0 , k = 1..r$$

r – hodnost

(1) $b \in R$

$$\hat{x} = A^{-} b$$

$$A\hat{x} = AA^{-} b \quad \stackrel{?}{=}$$

$$AA^{-} = \sum_k \sum_e s_k \frac{1}{s_e} |u_k\rangle \underbrace{\langle v_k | v_e\rangle}_{\delta_{ke}} \langle u_e|$$

$$= \sum_{k=1}^{r} |u_k\rangle\langle u_k| = S_R \quad \text{(projektor na range)}$$

$$\stackrel{?}{=} \quad S_R b = b$$

(2) $b \notin R$

$$\hat{x} = A^{-} b$$

$$A\hat{x} = S_R b \quad \text{(projekce pravé strany na range)}$$

Schmidtova ortogonalizace

$|a_1\rangle, |a_2\rangle, \dots, |a_N\rangle$

$|e_1\rangle \propto |a_1\rangle$

$|e_2\rangle \propto \left[1 - |e_1\rangle\langle e_1|\right] |a_2\rangle$

$\vdots$

$$|e_k\rangle \propto \left[1 - \sum_{l=1}^{k-1} |e_l\rangle\langle e_l|\right] |a_k\rangle$$

$\langle e_k|e_l\rangle = \delta_{kl}$

např. $\langle e_1|e_3\rangle = \langle e_1|a_3\rangle - \langle e_1|a_3\rangle = 0$.

$$\sum_{k=1}^{N} |e_k\rangle\langle e_k| = P_N = 1_N$$

projektor na podprostor všech $\sum_{k=1}^{N} \alpha_k |e_k\rangle$

SVD ortogonalizace

$|a_1\rangle, |a_2\rangle, \dots, |a_N\rangle$

$$A = \begin{pmatrix} \begin{pmatrix} \\ |a_1\rangle \\ \\ \end{pmatrix} & \begin{pmatrix} \\ |a_2\rangle \\ \\ \end{pmatrix} & \dots & \begin{pmatrix} \\ |a_N\rangle \\ \\ \end{pmatrix} \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \quad \dots, \quad |N\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$A|x\rangle = \sum_k |a_k\rangle\langle k|x\rangle = \sum_k x_k |a_k\rangle$$

tj. lin. obal sloupců $|a_k\rangle$ = range A

ale $A = \sum_k^r s_k |u_k\rangle\langle v_k| = U S V^\dagger$

ortog. báze range

$$U = \begin{pmatrix} \begin{pmatrix} |u_1\rangle \\ \parallel \\ |e_1\rangle \end{pmatrix} & \begin{pmatrix} |u_2\rangle \\ \parallel \\ |e_2\rangle \end{pmatrix} & \dots & \begin{pmatrix} |u_N\rangle \\ \parallel \\ |e_N\rangle \end{pmatrix} \end{pmatrix}$$

von Mises

$$A = \lambda_1 |x_1\rangle\langle x_1| + \lambda_2 |x_2\rangle\langle x_2|$$

$$|v_0\rangle = \alpha |x_1\rangle + \beta |x_2\rangle$$

$$\langle x_k | x_l \rangle = \delta_{kl} \, , \quad \alpha^2 + \beta^2 = 1$$

$$A|v_0\rangle = \alpha \lambda_1 |x_1\rangle + \beta \lambda_2 |x_2\rangle$$

$$|v_1\rangle = \frac{\alpha \lambda_1 |x_1\rangle + \beta \lambda_2 |x_2\rangle}{\sqrt{\alpha^2 \lambda_1^2 + \beta^2 \lambda_2^2}}$$

$$\alpha \rightarrow \frac{\alpha \lambda_1}{\sqrt{\alpha^2 \lambda_1^2 + \beta^2 \lambda_2^2}}$$

$$\frac{\lambda_1^2}{\alpha^2 \lambda_1^2 + \beta^2 \lambda_2^2} > 1$$

konvexní kombinace