Gaussova integrace

$$\int_a^b f(x)\,dx \approx \sum_{j=1}^{N} w_j f(x_j)$$

hledáme w_j a x_j, aby aprox. byla exaktní

pro polynomy

(a) polynomy $P_k(x)$, $k = 0, \dots, N$

ortogonalita: $\int_a^b P_k(x) P_\ell(x)\,dx = 0$, $k \neq \ell$

(b) kořeny polynomu $P_N(x)$

$$P_N(x_j) = 0 \ , \ j = 1, \dots, N$$

(c) váhy w_j řeší soustavu

$$\begin{pmatrix} P_0(x_1) & P_0(x_2) & \cdots & P_0(x_N) \\ P_1(x_1) & P_1(x_2) & \cdots & P_1(x_N) \\ & & \vdots & \\ P_{N-1}(x_1) & P_{N-1}(x_2) & \cdots & P_{N-1}(x_N) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} \int_a^b P_0(x)\,dx \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$P_0(x) \sim 1$

$$\int_a^b P_k(x)\,dx \sim \int_a^b P_k(x)\,P_0(x)\,dx = 0\,, \quad k \neq 0$$

a dále

$$\sum_j w_j \underbrace{P_N(x_j)}_{\text{kořeny}} = 0 = \int_a^b P_N(x)\,dx$$

tj: integrace je exaktní pro $P_0, P_1, \dots, P_N$

+ dá se ukázat i pro dalších $N-1$ polynomů,

celkem $2N-1$

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integrace s vahou

$$\int_a^b \underbrace{f(x)}_{\text{singularita}}\,dx = \int_a^b \underbrace{v(x)\,g(x)}_{\text{odstranění singularity}}\,dx \sim \sum_{j=1}^{N} w_j\, g(x_j)$$

např. $\displaystyle\int_{-1}^{1} \frac{e^{-\cos(x)^2}}{\sqrt{1-x^2}}\,dx \;\rightarrow\;$ volím $\displaystyle v(x) = \frac{1}{\sqrt{1-x^2}}$

Gaussova integrace $\rightarrow$ polynomy ortogonální s vahou

diferenciální rovnice

N-tý řád $\longrightarrow$ N rovnic 1. řádu

$$\frac{d^N y}{dx^N} = f\left(x, y, y', \ldots, \frac{d^{N-1} y}{dx^{N-1}}\right)$$

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_N &= \frac{d^{N-1} y}{dx^{N-1}} \end{aligned}\right\} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{N-1}' &= y_N \\ y_N' &\equiv \frac{d^N y}{dx^N} = f(x, y_1, y_2, \ldots, y_N) \end{aligned}$$

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např. $y'' + \alpha y' + \beta y = 0$

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\} \qquad \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -\alpha y_2 - \beta y_1 \end{aligned}}$$