# Modelování dat

(+ cvičení!)

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y(x_i; a_1, a_2, \dots a_M)}{\sigma_i} \right)^2 , \quad \nu = N - M$$

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přímka $\quad y(x) = ax + b$

$$\chi^2(a,b) = \sum_i \left( \frac{y_i - a x_i - b}{\sigma_i} \right)^2$$

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$$\frac{\partial \chi^2}{\partial a} = -2 \sum_i \frac{x_i}{\sigma_i^2} (y_i - a x_i - b) = 0$$

$$\frac{\partial \chi^2}{\partial b} = - \sum_i \frac{1}{\sigma_i^2} (y_i - a x_i - b) = 0$$

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$$\overbrace{\sum_i \frac{x_i y_i}{\sigma_i^2}}^{S_{xy}} - a \overbrace{\sum_i \frac{x_i^2}{\sigma_i^2}}^{S_{xx}} - b \overbrace{\sum_i \frac{x_i}{\sigma_i^2}}^{S_x} = 0$$

$$\underbrace{\sum_i \frac{y_i}{\sigma_i^2}}_{S_y} - a \underbrace{\sum_i \frac{x_i}{\sigma_i^2}}_{S_x} - b \underbrace{\sum_i \frac{1}{\sigma_i^2}}_{S} = 0$$

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$$S_{xy} = a S_{xx} + b S_x$$

$$S_y = a S_x + b S$$

$$\underbrace{\begin{pmatrix} S_{xx} & S_x \\ S_x & S \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} S_{xy} \\ S_y \end{pmatrix}$$

$$\det A = S_{xx} S - S_x^2 = \Delta$$

$$\det A_a = \begin{vmatrix} S_{xy} & S_x \\ S_y & S \end{vmatrix} = S_{xy} S - S_x S_y$$

$$\det A_b = \begin{vmatrix} S_{xx} & S_{xy} \\ S_x & S_y \end{vmatrix} = S_{xx} S_y - S_x S_{xy}$$

$$a = \frac{\det A_a}{\det A} = \frac{S_{xy} S - S_x S_y}{\Delta}$$

$$b = \frac{\det A_b}{\det A} = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta}$$

neurčitosti – primeri

$$\sigma_f^2 = \sum_i \sigma_i^2 \left(\frac{\partial f}{\partial y_i}\right)^2$$

$$a = \frac{S_{xy} S - S_x S_y}{\Delta}, \qquad b = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta}$$

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$$\frac{\partial a}{\partial y_i} = \frac{S\, x_i/\sigma_i^2 - S_x/\sigma_i^2}{\Delta} = \frac{S\, x_i - S_x}{\sigma_i^2 \Delta}$$

$$\sum_i \sigma_i^2 \frac{(S x_i - S_x)^2}{\sigma_i^4 \Delta^2} = \frac{1}{\Delta^2} \sum_i \frac{S^2 x_i^2 - 2 S S_x x_i + S_x^2}{\sigma_i^2}$$

$$= \frac{1}{\Delta^2}\left(S^2 S_{xx} - S S_x^2\right) = \frac{S}{\Delta} \,//$$

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$$\frac{\partial b}{\partial y_i} = \frac{S_{xx}/\sigma_i^2 - S_x\, x_i/\sigma_i^2}{\Delta} = \frac{S_{xx} - S_x x_i}{\sigma_i^2 \Delta}$$

$$\sum_i \sigma_i^2 \frac{(S_{xx} - S_x x_i)^2}{\sigma_i^4 \Delta^2} = \frac{1}{\Delta^2} \sum_i \frac{S_{xx}^2 - 2 S_{xx} S_x x_i + S_x^2 x_i^2}{\sigma_i^2}$$

$$= \frac{1}{\Delta^2}\left(S_{xx}^2 S - S_{xx} S_x^2\right) = \frac{S_{xx}}{\Delta} \,//$$

## zobecnený fit

$$\chi^2 = \sum_{i=1}^{N} \left[ \frac{y_i - \sum_k a_k X_k(x_i)}{\sigma_i} \right]^2$$

$$\frac{y_i}{\sigma_i} = b_i \quad , \quad \frac{X_k(x_i)}{\sigma_i} = A_{ik}$$

$$\chi^2 = \sum_i \left( b_i - \sum_k A_{ik} a_k \right)^2 = \| A\vec{a} - \vec{b} \|^2$$

$$\chi^2 = \left( \langle a | A^T - \langle b | \right) \left( A | a \rangle - | b \rangle \right)$$

$$\frac{\partial \chi^2}{\partial \langle a |} = A^T \left( A | a \rangle - | b \rangle \right) = 0$$

$$A^T A \, \vec{a} = A^T \vec{b}$$

$$\vec{a} = (A^T A)^{-1} A^T \vec{b}$$

$A : N \times M \; , \; N \geq M$

$A^T A : M \times M \; , \;$ poz. semidef.

# pseudoinverze

$A : N \times M$, $N > M$

$A^T : M \times N$

$A^TA : M \times M$

lin. nezávislé sloupce $A \Rightarrow A^TA$ regulární

$$A^- = (A^TA)^{-1}A^T$$

(1) $AA^-A = A(A^TA)^{-1}A^TA = A$

(2) $A^-AA^- = (A^TA)^{-1}A^TA(A^TA)^{-1}A^T = A^-$

(3) $(AA^-)^T = A^{-T}A^T = A(A^TA)^{-1}A^T = AA^-$

(4) $A^-A = (A^TA)^{-1}A^TA = \hat{1} \Rightarrow (A^-A)^T = A^-A$

Moore - Penrose podmínky

nevrčitosti obecně

$$\Gamma^a = J \Gamma^b J^T \quad , \quad J_{ki} = \frac{\partial a_k}{\partial b_i}$$

$$\Gamma^a_{kl} = \sum_i \sum_j \frac{\partial a_k}{\partial b_i} \Gamma^b_{ij} \frac{\partial a_l}{\partial b_j}$$

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$$a = A^- b$$

$$a_k = \sum_j (A^-)_{kj} b_j$$

$$\frac{\partial a_k}{\partial b_j} = (A^-)_{kj} \quad \Rightarrow \quad J = A^-$$

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$$\Gamma^b = \mathbb{1} \quad (\text{nezávislé, variance} = 1)$$

$$\Gamma^a = A^- A^{-T} = (A^T A)^{-1} A^T A \Big[\underbrace{(A^T A)^{-1}}_{\text{hermitovská}}\Big]^T$$

$$= (A^T A)^{-1} A^T A (A^T A)^{-1}$$

$$= (A^T A)^{-1}$$