L4

Fisherova informace

likelihood $\mathcal{L}(\theta) = P(D|\theta)$

F. I.

$$F = \left\langle \left(\frac{d}{d\theta} \log \mathcal{L}(\theta) \right)^2 \right\rangle$$

přes data

CRLB

nevychýlený odhad $\hat{\theta}$

$$\langle \hat{\theta} - \theta \rangle = 0$$

$$\boxed{(\Delta\theta)^2 \geq \frac{1}{F}}$$

saturace

- efektivní estimátor
- ML asymptoticky
- BLUE (GLS)

vychýlený odhad může být lepší!

OLS / GLS

$$A|x\rangle = |b\rangle + |\varepsilon\rangle = |d\rangle$$

šum — data

Gauss-Markov

$$\overline{|\varepsilon\rangle} = 0$$

$$\overline{|\varepsilon\rangle\langle\varepsilon|} = \sigma^2 \mathbb{1}$$

OLS

$$|\hat{x}\rangle = A^{-}|d\rangle \quad \ldots \text{BLUE}$$

1. linearita
2. unbiased

IC: $A^{+}A$ invertibilní A:

$$A^{-} = (A^{+}A)^{-1}A^{+}$$

např.

$$AA^{-}A = A(A^{+}A)^{-1}A^{+}A = A$$

atd.

tzn. $A^{-}A = (A^{\dagger}A)^{-1}A^{\dagger}A = 1$

$$\overline{|\hat{x}\rangle} = A^{-}\left(A|x\rangle + \overline{|\varepsilon\rangle}\right) = A^{-}A|x\rangle = |x\rangle$$

(3) biasd

$$|\hat{x}'\rangle = (A^{-} + B)\left(A|x\rangle + |\varepsilon\rangle\right)$$

$$\overline{|\hat{x}'\rangle} = A^{-}A|x\rangle + BA|x\rangle = |x\rangle + BA|x\rangle$$

$$\overline{|\hat{x}'\rangle} = |x\rangle \quad \forall |x\rangle \Rightarrow BA = 0$$

$$BA = 0 \Rightarrow A^{-}B^{\dagger} = (A^{\dagger}A)^{-1}\underbrace{A^{\dagger}B^{\dagger}}_{0} = 0$$

$$|\Delta x'\rangle = |\hat{x}'\rangle - |x\rangle = \left[(A^{-} + B)A - 1\right]|x\rangle + (A^{-} + B)|\varepsilon\rangle$$

$$= \left[\underbrace{A^{-}A - 1}_{0} + \underbrace{BA}_{0}\right]|x\rangle + (A^{-} + B)|\varepsilon\rangle$$

$$= (A^{-} + B)|\varepsilon\rangle$$

$$\overline{|\Delta x'\rangle\langle\Delta x'|} = (A^{-} + B)\overline{|\varepsilon\rangle\langle\varepsilon|}(A^{-\dagger} + B^{\dagger})$$

$$= \sigma^2\left(A^{-}A^{-\dagger} + \underbrace{A^{-}B^{\dagger}}_{0} + \underbrace{BA^{-\dagger}}_{0} + BB^{\dagger}\right)$$

$$\overline{|\Delta x'\rangle\langle\Delta x'|} = \sigma^2 A^- A^{-+} + \sigma^2 B B^+$$

$$\geq \sigma^2 A^- A^{-+} = \overline{|\Delta x\rangle\langle\Delta x|}_{OLS}$$

GLS

$$\overline{|\varepsilon\rangle\langle\varepsilon|} = \Gamma \quad , \quad \Gamma = C C^+$$

$$C^{-1} A |x\rangle = C^{-1}|b\rangle + \underbrace{C^{-1}|\varepsilon\rangle}_{|\varepsilon'\rangle} = C^{-1}|d\rangle = |d'\rangle$$

$$\overline{|\varepsilon'\rangle\langle\varepsilon'|} = C^{-1}\, \overline{|\varepsilon\rangle\langle\varepsilon|}\, C^{-1+} = C^{-1} C C^+ C^{-1+} = \mathbb{1}$$

GLS:

$$|\hat{x}\rangle = (C^{-1}A)^- |d'\rangle = (\underbrace{C^{-1}A}_{X})^- C^{-1}|d\rangle$$

$$X^- = (X^+ X)^{-1} X^+$$

$$\Rightarrow |\hat{x}\rangle = (A^+ C^{-1+} C^{-1} A)^{-1} A^+ C^{-1+} C^{-1} |d\rangle$$

$$|\hat{x}\rangle = (A^+ \Gamma^{-1} A)^{-1} A^+ \Gamma^{-1} |d\rangle \qquad \text{BLUE}$$

více parametrů

$$F_{kl} = \left\langle \left(\frac{\partial}{\partial \theta_k} \log \mathcal{L}(\theta) \right) \left(\frac{\partial}{\partial \theta_l} \log \mathcal{L}(\theta) \right) \right\rangle$$

CRLB pro kovarianční matici

$$\Gamma_{kl} = \langle \Delta\theta_k \Delta\theta_l \rangle$$

$$\boxed{\Gamma \geq F^{-1}}$$

Fish. matice

spec. případ – variance

$$\sigma_k^2 = \Gamma_{kk} \geq (F^{-1})_{kk}$$

alternativní forma F

$$F_{kl} = -\left\langle \frac{\partial^2}{\partial\theta_k \partial\theta_l} \log \mathcal{L}(\theta) \right\rangle$$

pouze pro dobře se chovající $\mathcal{L}(\theta)$

odvození :

$$-\frac{\partial^2}{\partial\theta_k \partial\theta_l}\log L(\theta) = -\frac{\partial}{\partial\theta_l}\left[\frac{1}{p(n|\theta)}\frac{\partial p(n|\theta)}{\partial\theta_k}\right]$$

$$= \frac{1}{p^2}\frac{\partial p}{\partial\theta_k}\frac{\partial p}{\partial\theta_l} - \frac{1}{p}\frac{\partial^2 p}{\partial\theta_l \partial\theta_k}$$

$$\left\langle \frac{1}{p^2}\frac{\partial p}{\partial\theta_k}\frac{\partial p}{\partial\theta_l}\right\rangle = \underbrace{\left\langle\left(\frac{\partial}{\partial\theta_k}\log p\right)\left(\frac{\partial}{\partial\theta_l}\log p\right)\right\rangle}_{F_{kl}}$$

$$\left\langle\frac{1}{p}\frac{\partial^2 p}{\partial\theta_l \partial\theta_k}\right\rangle = \int \frac{1}{\cancel{p(n|\theta)}}\frac{\partial^2 p(n|\theta)}{\partial\theta_l \partial\theta_k}\cancel{p(n|\theta)}\,dn$$

$$= \int \frac{\partial^2 p(n|\theta)}{\partial\theta_l \partial\theta_k}\,dn = \underbrace{\frac{\partial^2}{\partial\theta_l \partial\theta_k}\underbrace{\int p(n|\theta)\,dn}_{1}}_{0}$$

multinomiální statistika

$$\langle n_j \rangle = N p_j$$

$$\mathcal{L}(\theta) \propto \Pi_j \, p_j^{\,n_j}$$

$$\log \mathcal{L} \sim \sum_j n_j \log p_j$$

$$F_{kl} = \left\langle \sum_j \sum_{j'} \frac{n_j n_{j'}}{p_j p_{j'}} \frac{\partial p_j}{\partial \theta_k} \frac{\partial p_{j'}}{\partial \theta_l} \right\rangle$$

$$\langle \Delta n_j \Delta n_{j'} \rangle = N p_j (1 - p_j) \delta_{jj'} - N p_j p_{j'} (1 - \delta_{jj'})$$

$$= N p_j \delta_{jj'} - N p_j p_{j'}$$

$$\langle n_j n_{j'} \rangle = \langle \Delta n_j \Delta n_{j'} \rangle + \langle n_j \rangle \langle n_{j'} \rangle$$

$$F_{kl} = N \sum_j \frac{1}{p_j} \frac{\partial p_j}{\partial \theta_k} \frac{\partial p_j}{\partial \theta_l}$$

$$+ (N^2 - N) \underbrace{\sum_j \sum_{j'} \frac{\partial p_j}{\partial \theta_k} \frac{\partial p_{j'}}{\partial \theta_l}}_{0}$$

Poissonova statistika

$$\mathcal{L}(\theta) = \prod_j \frac{\langle n_j \rangle^{n_j}}{n_j!} e^{-\langle n_j \rangle}$$

$$\log \mathcal{L}(\theta) \sim \sum_j n_j \log \langle n_j \rangle - \sum_j \langle n_j \rangle$$

$$F_{kl} = \sum_j \sum_{j'} \frac{\langle n_j n_{j'} - \langle n_j \rangle \langle n_{j'} \rangle \rangle}{\langle n_j \rangle \langle n_{j'} \rangle} \frac{\partial \langle n_j \rangle}{\partial \theta_k} \frac{\partial \langle n_{j'} \rangle}{\partial \theta_l}$$

$$\langle n_j n_{j'} - \langle n_j \rangle \langle n_{j'} \rangle \rangle = \langle \Delta n_j \Delta n_{j'} \rangle = \langle n_j \rangle \delta_{jj'}$$

nezávislé pro $j \neq j'$

$$F_{kl} = \sum_j \frac{1}{\langle n_j \rangle} \frac{\partial \langle n_j \rangle}{\partial \theta_k} \frac{\partial \langle n_j \rangle}{\partial \theta_l}$$

pokud

$$\langle n_j \rangle = N p_j \;, \quad \sum_j p_j = 1$$

$$F_{kl} = N \sum_j \frac{1}{p_j} \frac{\partial p_j}{\partial \theta_k} \frac{\partial p_j}{\partial \theta_l}$$

normální statistika

$$\mathcal{L}(\theta) \propto \prod_j e^{-\frac{(n_j - \langle n_j \rangle)^2}{2\sigma_j^2}}$$

$$\log \mathcal{L}(\theta) \sim -\sum_j \frac{(n_j - \langle n_j \rangle)^2}{2\sigma_j^2}$$

$$F_{kl} = \sum_j \sum_{j'} \frac{\langle (n_j - \langle n_j \rangle)(n_{j'} - \langle n_{j'} \rangle) \rangle}{\sigma_j^2 \sigma_{j'}^2} \frac{\partial \langle n_j \rangle}{\partial \theta_k} \cdot \frac{\partial \langle n_{j'} \rangle}{\partial \theta_l}$$

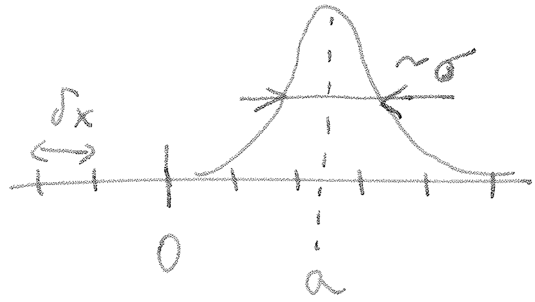
$$= \sum_j \frac{\langle (\Delta n_j)^2 \rangle}{\sigma_j^4} \frac{\partial \langle n_j \rangle}{\partial \theta_k} \frac{\partial \langle n_j \rangle}{\partial \theta_l}$$

$$F_{kl} = \sum_j \frac{1}{\sigma_j^2} \frac{\partial \langle n_j \rangle}{\partial \theta_k} \frac{\partial \langle n_j \rangle}{\partial \theta_l}$$

jednobodové rozlišení – astrometrie

Poisson. detekční šum

odhad parametru a



$$F = \sum_j \frac{1}{\langle n_j \rangle} \left(\frac{d \langle n_j \rangle}{da} \right)^2$$

Spojitá limita

$$\langle n_j \rangle \sim I(x_j)\, \delta x$$

$$F = \sum_j \frac{1}{I(x_j)} \left(\frac{d I(x_j)}{da} \right)^2 \delta x$$

$$\overset{\delta x \to 0}{\approx} \int_{-\infty}^{\infty} \frac{1}{I(x)} \left(\frac{d I(x)}{da} \right)^2 dx$$

Gauss. PSF

$$I(x) = \frac{N}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$\int I(x)\, dx = N \quad \ldots \text{ celk. intenzita}$$

6c

$$\frac{1}{I(x)}\left(\frac{d\,I(x)}{da}\right)^2 = \frac{N\,(a-x)^2}{\sqrt{2\pi}\,\sigma^5}\, e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} \downarrow \; dx = \frac{N}{\sigma^2}$$

$$\langle(\Delta a)^2\rangle \geq \frac{\sigma^2}{N}$$

celkový počet fotonů

kvantová tomografie

$$\rho \to \hat{\rho}$$

chyba

$$\varepsilon = \mathrm{Tr}\left[(\hat{\rho}-\rho)^2\right]$$ H-S vzdálenost

střední chyba

$$\bar{\varepsilon} = \left\langle \mathrm{Tr}\left[(\hat{\rho}-\rho)^2\right] \right\rangle$$

rozklad

$$\rho = \sum_k r_k \Gamma_k$$

$$\hat{\rho} = \sum_k \hat{r}_k \Gamma_k$$

$$\Downarrow$$

$$\varepsilon = \sum_k \sum_l (\hat{r}_k - r_k)(\hat{r}_l - r_l)\, \underbrace{\mathrm{Tr}(\Gamma_k \Gamma_l)}_{\delta_{kl}}$$

$$= \sum_k (\hat{r}_k - r_k)^2$$

$$\bar{\varepsilon} = \sum_k \left\langle (\hat{r}_k - r_k)^2 \right\rangle = \sum_k \sigma_k^2$$

$$\bar{\varepsilon} \geq \mathrm{Tr}(F^{-1})$$ CRLB

pro multinomiální rozdělení

$$F_{kl} = N \sum_j \frac{1}{p_j} \frac{\partial p_j}{\partial r_k} \frac{\partial p_j}{\partial r_l}$$

POVM

$$\Pi_j = \sum_k c_{jk} \Gamma_k$$

$$p_j = \mathrm{Tr}(\rho \Pi_j) = \sum_k c_{jk} r_k$$

$$\frac{\partial p_j}{\partial r_k} = \mathrm{Tr}(\Gamma_k \Pi_j) = c_{jk}$$

$$\Downarrow$$

$$F_{kl} = N \sum_j \frac{c_{jk} c_{jl}}{p_j}$$

$$C = [c_{jk}] \quad , \quad P = \begin{pmatrix} p_1 & p_2 & \dots \end{pmatrix}$$

$$F_{kl} = N \sum_j \frac{(C^{+})_{kj} c_{jl}}{p_j}$$

$$F_{\cancel{kk}} = N\left(C^+ P^{-1} C\right)$$

$$\bar{\varepsilon} \geq \frac{1}{N} \underbrace{\mathrm{Tr}\left[\left(C^+ P^{-1} C\right)^{-1}\right]}_{\text{chyba pro jednu kopii}}$$

měření 1 : $\{\Pi_j\}_1$, $j = 1 \ldots M$

měření 2 : $\{\Pi_j\}_2$, $j = 1 \ldots M$

měření : $p_1 \{\Pi_j\}_1 + \underset{p_2}{\underset{\shortparallel}{(1-p_1)}} \{\Pi_j\}_2$

$$F = p_1 F_1 + p_2 F_2$$

funkce $1/x$

$$F^{-1} \leq p_1 F_1^{-1} + p_2 F_2^{-1}$$

středování přes unitárně ekvivalentní stavy

$\overline{F^{-1}} \leq \overline{F_1^{-1}} = \overline{F_2^{-1}}$, pokud $\cancel{F}$ $\{\Pi_j\}_1$ a $\{\Pi_j\}_2$

jsou unitárně ekv. => optimalita kovariantních měření

Superoperátory

$$\rho \to \begin{pmatrix} r_1 \\ r_2 \\ \vdots \end{pmatrix} \equiv |\rho\rangle\rangle$$

$$F = \sum_j \frac{|\Pi_j\rangle\rangle\langle\langle\Pi_j|}{p_j(\rho)}, \quad Tr(\Gamma_k) = 0, \ \forall k$$

popř.

$$F = \sum_j \frac{|\bar{\Pi}_j\rangle\rangle\langle\langle\bar{\Pi}_j|}{p_j(\rho)} \quad \text{v l.i.s. bázi}$$

kde

$$\bar{\Pi}_j = \Pi_j - \frac{1}{d} Tr(\Pi_j)$$

je projekce Π_j do podprostoru bezestopých op.

kovariantní měření

$$F = d \int \frac{1}{\langle\psi|\rho|\psi\rangle} |\bar{\Pi}_\psi\rangle\rangle\langle\langle\bar{\Pi}_\psi| \, d\mu(\psi)$$

$Tr(F^{-1}) = 2(d-1)$ pro čistý a $d^2 + d - 1 - \frac{1}{d}$ pro max. smíš.