music theorists. Their work was often inconsistent, however, and they came to conflicting conclusions. This project has given an objective and consistent evaluation over a wide range of parameters, and a meaningful measure of just how good a fit various equal temperaments can provide to the important just intervals. The familiar case $N = 12$ is found to be less remarkable than is commonly believed, and the other best fits found might be characterized as only moderately remarkable, having occurred in spite of odds on the order of ten-to-one against.

A final question raised by Hartmann is whether there is any way to "derive" $N = 12$. While I do not think it should be given the status of a proof, there is an algorithm studied by Brun[18] that generates a series beginning with $N = 12$, 53, 359,... when applied to y_{32}. I have concluded that this procedure is at best "suggestive" of important values of N when it is generalized to include other intervals; further comments may be found in Ref. 17.

In the past one might have dismissed this whole investigation as being of purely academic interest, because of the impracticability of performing music on traditional instruments built to embody very large values of N. However, we live in an era when computer-generated sound and computer-assisted instruments might well use large-N equal temperaments. This work could be of some use in giving a clear idea just how much can be accomplished with various choices of N.

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[10]The most complete summary of classic work is in the unpublished dissertation by Joel Mandelbaum, "Multiple Division of the Octave and the Tonal Resources of 19-Tone Temperament" (Indiana University, Bloomington, IN, 1961), especially Chap. 13.
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[14]To further justify dismissing $N = 65$, 94, 106, etc. as merely "riding on the coattails" of earlier cases, note the following general property: If y_{ij} is very nearly an integer multiple of both $1200/N_1$ and $1200/N_2$, then it must also be close to a multiple of $1200/(N_1 + N_2)$.

[15]One objection to the present procedure is that it treats all p_{ij} in each case as equally important, whereas it is really easier to judge the mistuning of some intervals than others. Indeed, Helmholtz and others have proposed various versions of a weighting factor called "consonance rank." Recent experiments [D. Hall and J. Hess, Music Percept. **2**, 37 (1984)] provide empirical data about such judgments, and it would be more important to take this into account if using an acoustical criterion associated with e_{ij} instead of p_{ij}. But in order to work out the present probabilistic view I believe the simplicity of equal weights is worth retaining.

[16]One should hesitate to push on indefinitely to include larger and larger prime factors. Very high harmonics will go beyond the audible range. Even more to the point, for all real musical instruments the strength of the spectral components drops off rapidly with harmonic number, leaving only very weak beats to provide clues to the mistuning of complex intervals. But neither should such intervals be dismissed offhand, for the experiment of Hall and Hess (see Ref. 15 above) showed that some listeners under favorable conditions can make judgments on mistuning of intervals involving prime numbers 7, 11, and 13.

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Aberration and Doppler shift: An uncommon way to relativity

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Aberration, as well as the Doppler shift, are phenomena close to everyday experience. Special aspects of relativity are derived from axioms concerning the conformal celestial spheres of coinciding observers and are presented with elementary geometrical tools. One way of linking the mathematical hypotheses to physical quantities related to space-time is demonstrated. The classical formulas for aberration and Doppler shift are used as starting points for deriving the relativistic counterparts.

I. INTRODUCTION

The special theory of relativity can be derived from several seemingly equivalent systems of hypotheses. Some of them use the constancy of the velocity of light as an axiom, others derive it as a consequence of other axioms.[1–5] It is even possible to use a Lorentz invariant vacuum medium, say ether, to arrive at the special theory of relativity, although "the ether just fades away"[6] by exploiting the axioms.[7,8] Ether is not contradictory to relativity, but it is not necessary.

Confusion was caused for a long time by the different interpretations commonly used for the words "observer" and "observe."[9,10] The first operational concept of the "observer" was introduced by Einstein in his 1905 paper on "Die Elektrodynamik bewegter Körper." This type of observer collects data on events in space and time with a whole set of recording clocks evenly distributed in space. The clocks for this Einstein type observer are associated with an inertial frame of reference. This concept is useful for physicists in their technical work, but is a difficult introduction to the theory. A second concept of the relativistic "observer" is close to the everyday use of the word. This observer is located at one place in space and collects all information by the light arriving at his position.

This second type of observer is confronted with phenomena like aberration and Doppler shift whereas the Einsteinian observer registers time dilation and Lorentz contraction.

When we use the word observer in the rest of this article, we always mean an individual observer who collects all information from the retarded light cone. He registers the light arriving at his location from all directions of his celestial sphere.

Many textbooks discuss at length the complicated and somehow impossible thought experiments with astronaut twins and trains to explain the Lorentz contraction and time dilation. Being one-way experiments, they are contradictory to relativity and sometimes ignore the question, "what is the same physical quantity for different observers?"[11]

It might be more reasonable to explain some aspects of special relativity by using observations that can be understood by feasible, direct experience in everyday life: Doppler shift and aberration. Both effects are caused by the finite velocity of any information transfer and can be demonstrated with either sound waves or light. Doppler shift is easily witnessed near any street or railway line with fast moving vehicles and aberration with fast moving airplanes, where the sound seems to come from a direction somewhere behind the visible plane.

Aberration and Doppler shift allow one to describe a good portion of the special theory of relativity. It will be shown how it is even possible to reconstruct this part of the theory by introducing a system of hypotheses solely based on observation and comparisons of the celestial spheres of coinciding observers of the second type.

However, this observer cannot be pointlike, as he is treated in the following parts of this article, and has to be considered as an idealization. Every gain of information needs a finite amount of time to elapse and occurs in a more or less extended spatial dimension. This concept of localized observer must be dropped as soon as one deals with other physical quantities than aberration, dynamics needs a space-time concept.

II. ABERRATION

Most of the classical formulas look simpler than their relativistic counterparts. However, exploitation of the formulas often reveals simpler laws and higher symmetries in the relativistic case. Aberration of light is an apparent shift of the direction of incident beams due to the motion of the observer relative to the light source, and offers a nice example for demonstrating this aspect. Einstein already presented the equation for the relativistic aberration of light in his 1905 paper. The equation in a different and little used

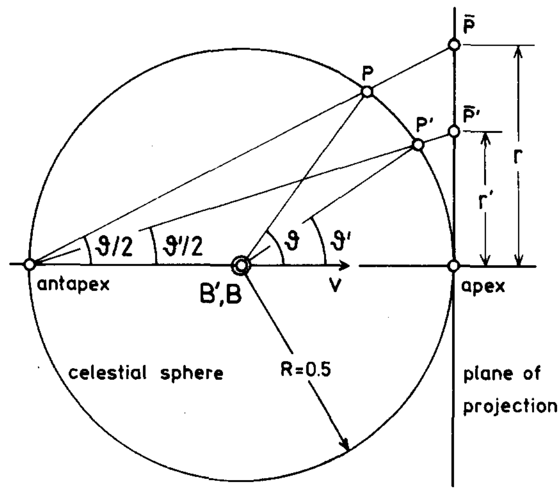

Fig. 1. Stereographic projection of the celestial spheres illustrating the angles ϑ and ϑ' between the direction of the relative motion and the incident light beam as seen by coinciding observers.

form[12] is

$$\tan(\vartheta/2) = [(1+\beta)/(1-\beta)]^{1/2}\tan(\vartheta'/2)\,, \qquad (1)$$

where ϑ and ϑ', respectively, represent the angles between the direction of the relative motion and the incident light as observed by the two observers and $\beta = v/c$ is the relative normalized velocity of these observers. This form has the advantage of a simple interpretation of the aberration. Equation (1) describes a bijective mapping between the celestial spheres of the two coinciding observers (Fig. 1):

$$(\vartheta,\phi) \rightarrow (\vartheta',\phi' = \phi), \qquad (2)$$

where the ϑ and ϕ define spherical coordinates. The mappings between celestial spheres are conformal on the whole spheres; this means the corresponding angles are equivalent and circles map to circles. This fact was used to show that spheres present circular outlines to every observer, independent of their relative movement.[12]

The above-stated qualities of the mapping can be proven with elementary geometrical tools. Due to the cylindrical symmetry defined by the axis of the relative movement of the two observers, the mapping of the spheres has two diametrical fixed points called apex and antapex of the motions. To discuss the mapping more easily, the celestial spheres are projected stereographically onto a plane (Fig. 1). Antapex is used as the projection center. This projection of the spheres onto a plane is unique, reversible, and conformal. Therefore, once again circles map to circles. On the plane, Eq. (1) relates $\tan(\vartheta/2)$ and $\tan(\vartheta'/2)$ to the respective distances r and r' from the image of the apex (Fig. 1):

$$r = \tan(\vartheta/2) \text{ and } r' = \tan(\vartheta'/2). \qquad (3)$$

Thus with Eq. (1) the above gives

$$r = A(\beta)r', \qquad (4)$$

which describes a dilation since $A(\beta)$ is a positive number. Because a dilation is conformal, we have proven that the aberration formula (1) defines a conformal mapping on the celestial spheres.

The set of dilations of a plain with a common center forms an algebraic group under composition. Correspondingly, the set of conformal mappings of the celestial spheres that obey Eq. (1) also form a group under composition. These mappings are related to different pairs of coinciding observers which all have collinear relative motions.

The group quality of the mappings described by the aberration formula (1) now leads to the addition law for collinear velocities:

$$\tan(\vartheta'/2) = A(\beta_2)\tan(\vartheta''/2)$$
$$\tan(\vartheta/2) = A(\beta_1)\tan(\vartheta'/2) \quad (5)$$

can be converted into

$$\tan(\vartheta/2) = A(\beta_3)\tan(\vartheta''/2) \quad (6)$$

with

$$\beta_3 = (\beta_1 + \beta_2)/(1 + \beta_1\beta_2), \quad (7)$$

which is the well-known relation.

Now, the classical aberration equation shall be discussed for comparison with the relativistic version. The aberration of light was discovered by Bradley in 1729. This discovery brought the general accordance that light is traveling at a finite speed. The classical formula reads

$$\tan\vartheta = \sin\vartheta'/(\cos\vartheta' - \beta), \quad (8)$$

which is simpler than the relativistic counterpart [compare with Eqs. (13) and (14)]. This equation also describes a mapping of the celestical spheres of coinciding observers. Except for the cylindrical symmetry this mapping cannot be described by simple symmetries and conservation laws. It is not conformal and it does not take circles into circles, which is easily demonstrated by choosing an example.

The velocity of light plays no special role in the classical theory. Therefore, Eq. (8) is most easily interpreted as a mapping of celestial spheres, if observer B′ who sees angles ϑ' is at rest relative to the light source considered. Then, $\beta = v/c$ is composed of the relative velocity v of the observers and the velocity of light in the rest frame of the light source. For any other couple of coinciding observers the apparent velocity of light becomes dependent on the velocity of B′ relative to the light source. As a consequence, the parameter $\beta = v/c'$ now contains a velocity of light c', which is also dependent on ϑ'. This makes an interpretation of Eq. (8) as a set of mappings of the celestial spheres rather awkward.

III. RECONSTRUCTING THE THEORY

The mentioned qualities of the relativistic aberration formula (1) now suggest a new possible way of reconstructing the special theory of relativity. This is done by using hypotheses exclusively to describe and compare the celestial spheres of coinciding observers. This was proposed by A. Komar[13] in 1965 in his article "Foundations of Special Relativity and the Shape of the Big Dipper" in this Journal. Komar motivated his idea with the mathematical theorem: "The algebraic group of conformal mappings of a 2-dimensional sphere onto itself is homomorphic to the group of the homogeneous orthochronous Lorentz group."

The suggested hypotheses are[14]: (a) given two observers who instantaneously coincide, the celestial spheres which each observes at that instant provides a legitimate representation of the same physical situation and (b) the celestial spheres of the two observers are conformal.

As Komar points out, "the first statement of the principle of relativity stems from the recognition that coinciding observers collect information from the same retarded light cone, regardless of their relative state of motion. It therefore incorporates some of the content of the hypothesis of the constancy of the velocity of light."

In this article, it shall be shown how some of the statements of special relativity can be derived from the above-stated hypotheses by using elementary arguments. Much care has to be taken not to use hidden assumptions and to clearly state all assumptions used. This is especially important since the result of the attempt is already known. It is very easy to get trapped with right results but wrong logic.

Primarily, nothing is specified about how the coincidence is characterized and what distinguishes each observers' motions. Most generally, the relative motion can be described by scalar, vectorial, or tensorial quantities. First a restriction concerning the specification of the coincidence and thus of the class of equivalent observers must be made: The coincidence of the two observers distinguishes only one direction on the celestial spheres. All vectorial quantities that define the relative motion are collinear.

To make sure that no other directions are distinguished otherwise, one more hypothesis is necessary. Since the first two axioms only make statements about the comparison of the celestial spheres of coincident observers, one more assumption about the celestial sphere of one single observer is needed: (c) the celestial sphere of one single observer is isotropic.

The restriction of the character of the coincidence and the third hypothesis define a cylindrical symmetry for the aberration. Now, the required conformal mapping must have two diametrical fixed points which we again may call apex and antapex. As discussed in Sec. II, the celestial spheres may be projected stereographically onto a plane (Fig. 1), again using antapex as the projection center. The conformal mapping of the celestial spheres is thus projected into a conformal mapping of the corresponding planes. The image of apex remains a proper fixed point whereas the antapex itself is projected to infinity. Thus the antapex becomes an improper fixed point. The projection plane now can be treated as the complex plane.

A linear fractional transformation, often called a Möbius transformation:

$$w = f(z) = (az + b)/(cz + d), \quad ad - bc \neq 0, \quad (9)$$

with the complex constants a, b, c, and d and the complex variables z and w is conformal on the complex plane except for $z = -d/c$, for which point the denominator of (9) becomes zero. The origin $z = 0$ of the complex plane can arbitrarily be chosen to be the proper fixed point of the mapping; it shall be the projection of the apex. This is equivalent to $f(0) = 0$ and yields $b = 0$. If we consider the infinitely remote point is an improper fixed point, no other point of the complex plane shall be be mapped to infinity and therefore $c = 0$. With these results we arrive at a mapping

$$w = f(z) = (a/d)z = Az, \quad A \neq 0, \quad (10)$$

which is now conformal and reversible on the whole complex plane, as required by hypothesis (a). The same result could be obtained by using the Liouville theorem from complex analysis which implies that the only mapping conformal on the whole complex plane is a linear transformation. The cylindrical symmetry of the conformal mapping on the spheres demands that A is a positive real number and the mapping (10) becomes a dilation with the center $z = 0$. Corresponding to Eq. (3), the mapping on the celestial spheres now obeys an equation

$$\tan(\vartheta/2) = A\tan(\vartheta'/2)\,, \quad (11)$$

which is essentially the statement of Eq. (1). The dimensionless scalar quantity A characterizes the one-dimensional relative motion.

IV. VELOCITY

At this point, the hypotheses are essentially exhausted. To arrive at the aberration in Eq. (1), one needs some link between the mathematical ideas of the hypotheses and operationally defined physical quantities.

It is reasonable to begin with the most basic parameters used in classical physics to describe motion: velocity and acceleration. The question arising at this stage is how parameter A is connected with velocity, acceleration, and possibly higher derivatives. The above-stated hypotheses (a)–(c) do not give any information on this.

Instead of introducing a new hypothesis on this connection, a new method is proposed here. Since classical physics is a very good description for many phenomena in our world, it should stay valid within a certain limit ($\beta \to 0$) of the new theory.

In Eq. (4) only one parameter A defines the conformity of the celestial spheres. This one parameter can be substituted by another parameter β:

$$A = A(\beta)\,. \quad (12)$$

To be safe from using any hidden assumptions about other quantities also characterizing the coincidence of the observers, one more restriction about the class of relative motion to be considered is introduced: The relative motion of coinciding observers is defined by one single vectorial quantity.

For choosing a relation $A(\beta)$ that can be related to classical velocity, the following idea shall be exploited: As described in Sec. II, the classical aberration formula (8) does not describe a conformal mapping on the celestial spheres and thus contradicts hypothesis (b). As a valid limit of the conformal aberration, it is assumed that the classical aberration can be corrected by an appropriate factor $B(\beta)$:

$$\tan\vartheta = B\sin\vartheta'/(\cos\vartheta' - \beta)\,. \quad (13)$$

A straightforward calculation transforming Eq. (13) into the form of Eq. (11) yields

$$B = (1-\beta^2)^{1/2} = 1/\gamma \quad (14)$$

and thus

$$A(\beta) = [(1+\beta)/(1-\beta)]^{1/2}\,, \quad (15)$$

which fulfills the above-stated requirement of the classical limit $\beta \to 0$. With this result, we again arrive at the aberration formula (1).

V. DOPPLER SHIFT

Thus far, no time and distance measurement and consequently, no space-time concept has been introduced, since a coincidence of observers occurs in one point of space-time. At this point, no definition of time measurements shall be discussed. But it is interesting to note in this context that, similar to aberration, a heuristic derivation of the Doppler shift formula is possible. Using the Galilean principle of relativity for deriving the classical formula for the Doppler shift of light yields

$$\omega' = \omega(1-\beta\cos\vartheta) \quad (16a)$$

$$\omega = \omega'(1+\beta'\cos\vartheta')\,, \quad (16b)$$

but also requires $\vartheta = \vartheta'$ and $c' = c(1-\beta\cos\vartheta)$.[14] This contradicts the above-stated aberration and the constancy of the velocity of light.

To regain the conformal aberration (1), Eqs. (16a) and (16b) can also be corrected by a factor D:

$$\omega' = \omega D(1-\beta\cos\vartheta) \quad (17a)$$

$$\omega = \omega' D(1+\beta\cos\vartheta')\,. \quad (17b)$$

By substituting (17a) in (17b) and applying the aberration formula in the yet different form

$$\cos\vartheta = (\beta + \cos\vartheta')/(1+\beta\cos\vartheta')\,, \quad (18)$$

one can calculate D as

$$D = \gamma = (1-\beta^2)^{-1/2}\,. \quad (19)$$

The relativistic Doppler shift formula thus is

$$\omega' = \omega\gamma(1-\beta\cos\vartheta) \quad (20)$$

or using (18) it becomes

$$\omega' = \omega\gamma[1-\beta(\beta+\cos\vartheta')/(1+\beta\cos\vartheta')]\,. \quad (21)$$

Equations (20) and (21) allow us to discuss Doppler shift and its relation to aberration and time measurement.

For a movement in the direction towards the light source, which means $\vartheta = \vartheta' = 0$, Eq. (21) becomes

$$\omega' = \omega\gamma(1-\beta)\,. \quad (22)$$

The second term in the bracket of Eq. (22) means the pure radial velocity. Equation (20) can be interpreted geometrically as illustrated in Fig. 2. Observer B shall be resting relative to the light source L and observer B′ moves with the velocity β in the indicated direction. Due to aberration, the angle ϑ' between the apparent direction towards the light source and the direction of the relative movement as observed by B′ is different to the corresponding angle ϑ as observed by B. Thus observer B′ registers a Doppler shift which corresponds to his radial velocity relative to the light source, but to his radial velocity as measured in the frame associated with observer B.

The factor γ in Eq. (20) can be interpreted as a consequence of the different time scales in the two frames B and B′. For $\vartheta = 90°$, i.e., for a pure transverse motion of B′ as observed in the rest frame B of the light source L, the Doppler shift as observed by B′ is a pure consequence of time dilation:

$$\omega' = \omega\gamma. \quad (23)$$

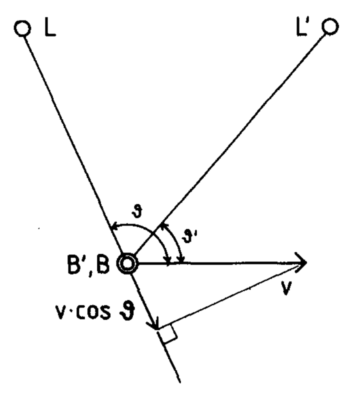

Fig. 2. Coinciding observers viewing the same light source L with respect to L' in the directions ϑ with respect to ϑ'. The observer B rests with respect to the light source.

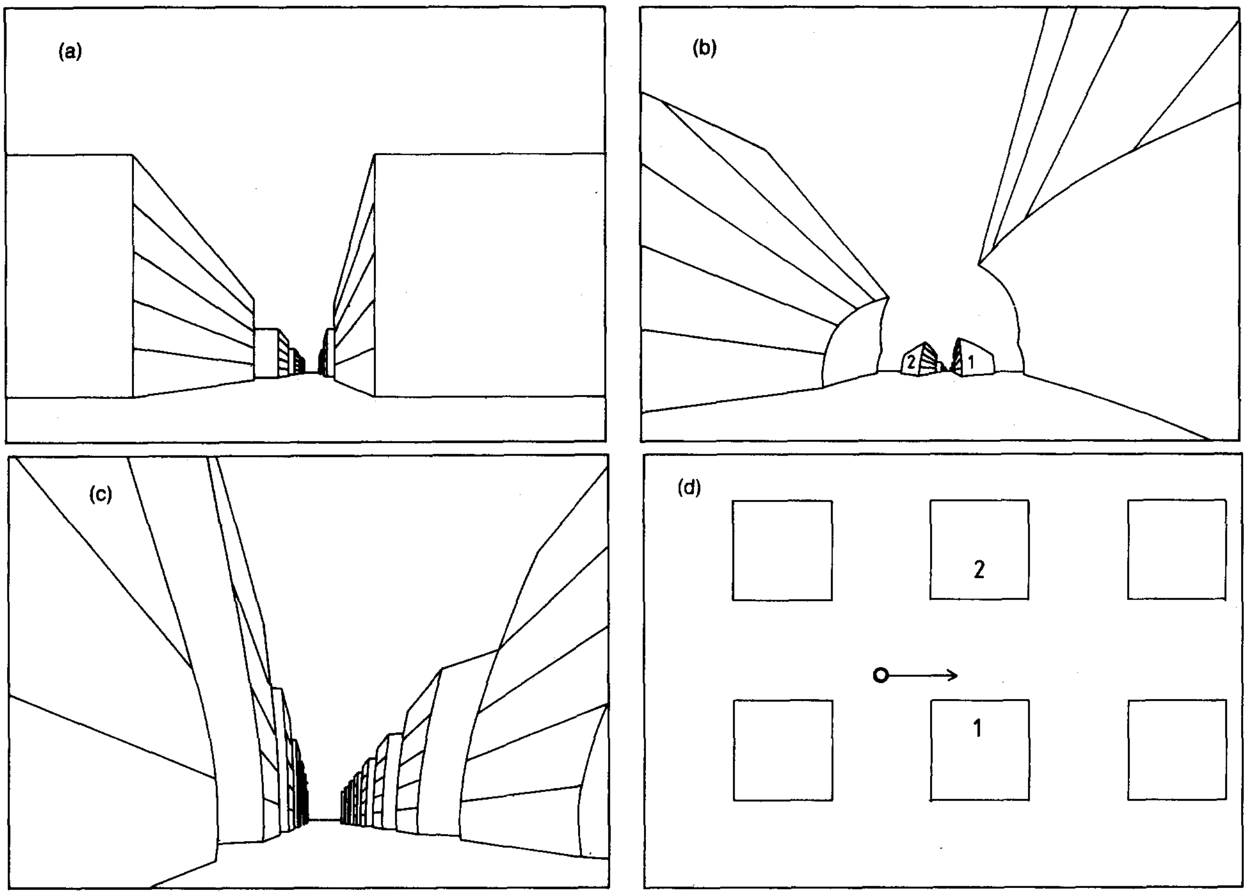

Fig. 3. The city street as viewed by Mr. Tompkins riding on his bicycle along the lane: (a) nonrelativistic view, (b) relativistic view with a speed $\beta = 0.8$ in Relativity Land, looking forward and (c) looking backward at the same speed, (d) street map with the house numbers, Mr. Tompkins' position, and his direction of motion.

This suggests a possibility to observe a proper time dilation and is deduced as a consequence of the aberration.

For an apparent pure transverse motion of B′ relative to the source, at $\vartheta' = 90°$, the light still appears Doppler shifted:

$$\omega' = \omega/\gamma . \tag{24}$$

This transverse Doppler shift can be interpreted as a consequence of time dilation and aberration. If the apparent movement is purely transverse for B′, this motion is not purely transverse for B and the term $\beta \cos \vartheta$ in Eq. (20) is the radial component of the velocity β. Therefore, according to Eq. (20), B′ observes a relativistic Doppler shift due to an apparent radial velocity in frame B but in a direction $\vartheta' = 90°$ due to aberration in frame B′.

VI. DISCUSSION

In the derivations described there was no need to introduce an operational concept of space and time. The coincidence used in the original hypotheses was considered to occur at only one point of space-time. With an operational time measurement by counting cycles of a stable oscillator, the constancy of the velocity of light also allows an operational concept of distance measurements. To be safe with the time concept, identical oscillators are allowed to move with each observer who then measures its proper time.

One is thus able to measure distances from the observer to some point in space by radar sounding, although for practical reasons only within interplanetary distances. The advantage of this radar method is the exclusive use of two-way experiments that are the operational experiments in relativity theory.

In this way, each uniformly moving observer defines his own local space-time. What is interesting, is how these frames of different observers are transformed into each other. Here, a derivation of the Lorentz transformation that manages this is needed.

With the derived formulas (1) and (20), we arrive at a possible link between aberration and Doppler shift on one side and the Lorentz transformation on the other side. As mentioned in textbooks,[14] the formulas for relativistic aberration (1) and the Doppler shift (20) are equivalent to a Lorentz transformation of a four-vector $(k_0,\mathbf{k})$ with the components $k_0 = \omega/c$ and the wave vector $\mathbf{k}$ for light waves.

To use this approach as a first introduction to special relativity theory is not recommended. The usual way of deriving the Lorentz transformation and thus the formula for aberration is still a good approach to relativity. However, it is not necessary to exploit all consequences of difficult phenomena like the Lorentz contraction, time dilation, and the related so-called paradoxes at an early stage of teaching relativity. Aberration and also the Doppler shift offer a means of explaining a relativistic world far closer to imagination as well as reality. The twin paradox just ceases to be a difficult story when radio signals going forth and back transmit the information of the clock-time with the inevitable time lag.[15]

In G. Gamows' story "Mr. Tompkins in Wonderland,"[16] pictures have to be changed. The relativistic city street looks very strange to us due to aberration (Fig. 3) as was realized and described by several authors[9,17-21] after 1959. Specifically Roger Penrose [12] showed how a relativistically moving sphere displays a circular outline to any observer. Due to the Doppler shift, a color picture would look even more exotic, especially if we also take the relativistic change of light intensity depending on the relative motion of the light source into account.

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Semiclassical treatment of the double well

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The double well potential $V(x) = \frac{1}{4}\lambda(x^2 - \alpha^2)^2$ is addressed using both semiclassical path integral and instanton techniques. The basic physics of the two-state system is shown to arise, and energy levels calculated via the two methods are compared.

I. INTRODUCTION

The double well potential, as typified by the potential (cf. Fig. 1)

$$V(x) = \tfrac{1}{4}\lambda(x^2 - \alpha^2)^2, \tag{1}$$

offers a soluble example of a two-state system. As such, the physics is of great importance and is generalizable to situations far beyond the simple potential well indicated above—e.g., the ammonia molecule, the K^0–$\bar{K}^0$ system, etc.[1]

The physics of the double well is usually explored via the semiclassical WKB approximation solution to Eq. (1). This procedure is well known and we shall merely summarize the results:[2]

(1) The energy levels, as defined via the Schrödinger equation (note we are using $\hbar = 1$),

$$\left(-\frac{1}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi_n(x) = E_n\psi_n(x), \tag{2}$$

consist of a series of nearby pairs with

$$E_n^{\pm} = (n + \tfrac{1}{2})\omega_0 \pm (\omega_0/2\pi)\exp(-\tfrac{1}{2}W_2),$$
$$n = 0,1,2,\ldots. \tag{3}$$

Here,

$$\omega_0^2 = (1/m)V''(\alpha) = (2/m)\lambda\alpha^2 \tag{4}$$

is the classical oscillation frequency of either of the two wells and

$$\frac{1}{2}W_2 = \int_{-a}^{a} dx\sqrt{2m[V(x) - E]} \tag{5}$$

is the usual WKB penetration integral.

(2) The wavefunctions corresponding to these states are given approximately by

$$\psi_n^{\pm} = \sqrt{\tfrac{1}{2}}[\phi_n(x) \mp \phi_n(-x)], \tag{6}$$

where $\phi_n(x)$ is related to the single oscillator wavefunction $\chi_n(x)$ via

$$\left(-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2x^2\right)\chi_n(x) = \left(n + \frac{1}{2}\right)\omega_0\chi_n(x), \tag{7a}$$

with

$$\phi_n(x) = \chi_n(x - \alpha). \tag{7b}$$

(3) A particle, which at time $t = 0$ is on one side of the barrier, oscillates back and forth between the two wells with period

$$\tau = (\pi/\omega_0)\exp(\tfrac{1}{2}W_2). \tag{8}$$

While these results are straightforwardly derivable using WKB techniques,[2] it is interesting to note that they may also be obtained through two very different methods—the semiclassical path integral and the instanton procedure. It is our purpose here to outline these alternate techniques