Relativistic energy of a moving spherical capacitor

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Abstract
We discuss the relativistic transformation of the energy of a charged spherical capacitor. The energy stored in the electromagnetic fields observed by an uniformly moving frame is related to that of the rest frame by a factor different from γ. Considering the energy and the stresses in the capacitor shells, we show that the total energy of the system satisfies the expected relativistic transformation.

1. Introduction

A very interesting example of an apparent paradox in the relativistic transformation of the energy was given by Rindler and Denur [1] some time ago. They considered a parallel-plate capacitor as seen from a frame moving in the direction of the electric field. The electrostatic energy of the field configuration is reduced by a factor of $1/\gamma$ with respect to the energy at the rest frame. This seems to contrast with the expected relativistic result of the energy increasing by γ. However, taking into account the stress in the braces that keep the plates at fixed positions, they found the appropriate relativistic transformation for the total energy of the system. This is a very nice example of the role played by the tension in the relativistic dynamics of extended bodies.

In the case of the parallel-plate capacitor, only electric fields show up in both frames and the geometry is unchanged by coordinate transformation. When we consider a spherical capacitor, the change of observer has non-trivial effects in the geometry and in the field configuration. The spherical symmetry of the rest frame is not present for a moving observer. Furthermore, the field configuration for any moving frame will involve magnetic fields in addition to electric fields. Another important difference is that the relevant stress tensor contributions are distributed along the spherical capacitor shells while in the parallel-plate case they were acting just on the braces. We will see that the electromagnetic energy will transform with a factor different from γ but also different from the parallel-plate case. An
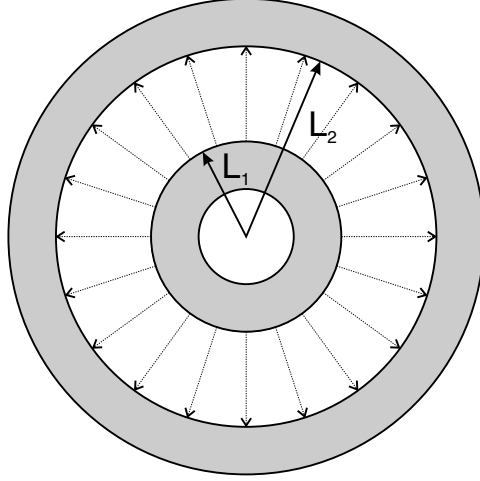
Figure 1. Capacitor in the rest frame S_0.
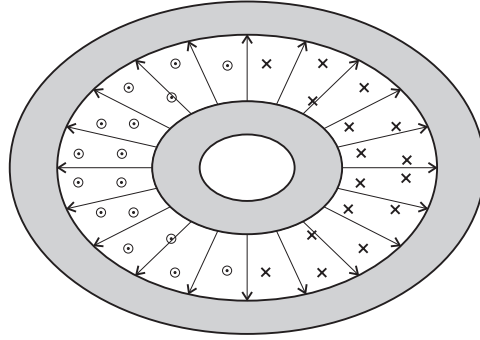
Figure 2. Capacitor moving upwards in the frame S with electric and magnetic fields. The magnetic field lines are circles around the direction of motion.

interesting aspect that will emerge is that the tensions inside the capacitor shells are not uniquely determined. Only their boundary values are fixed by the electric forces on the charge distribution. However, the transformation of the total energy of the system depends only on these boundary values.

2. Electromagnetic energy

Let us consider a spherical vacuum capacitor of internal radius L_1 and external radius L_2, with charge q on the inside shell and $-q$ on the outside, as shown in figures 1 and 2. S_0 is the rest frame of the capacitor and S is a frame moving with constant velocity $-v\hat{z}_0$ with respect to S_0. The coordinate systems on the frames are parallel and the origins coincide with the centre of the capacitor at $t_0 = t = 0$. While for S_0 the capacitor shells are spheres with the centre at coordinate origin, for S, by Lorentz contraction, they are ellipsoids centred at $vt\hat{z}$ and with semi-axis $L_1(1, 1, \gamma^{-1})$ and $L_2(1, 1, \gamma^{-1})$ in the (x, y, z) directions, respectively.

For the S_0 observer, the fields and the electromagnetic energy between the shells are just

$$\vec{E}_0 = \frac{1}{4\pi\epsilon_0}\frac{q\hat{r}}{r^2}, \qquad \vec{B}_0 = 0, \qquad U_{0_{EM}} = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{L_1} - \frac{1}{L_2}\right), \tag{1}$$

while for S the fields, obtained from the relativistic transformations, are

$$
\begin{aligned}
\vec{E} &= \left(\frac{q\gamma}{4\pi\epsilon_0}\right)\frac{x\hat{x} + y\hat{y} + (z - vt)\hat{z}}{(x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}}, \\
\vec{B} &= \left(-\frac{q\gamma v}{4\pi c^2\epsilon_0}\right)\frac{y\hat{x} - x\hat{y}}{(x^2 + y^2 + \gamma^2(z - vt)^2)^{3/2}}.
\end{aligned}
\tag{2}
$$

The electromagnetic energy involves integrating over the region between shells. This can be done by introducing an ellipsoidal coordinate system: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = \gamma^{-1}r\cos\theta + vt$. The energy density in these coordinates is

$$u_{EM} = u_{EM}(r, \theta) = \frac{\epsilon_0}{2}\left(\frac{q\gamma}{4\pi\epsilon_0}\right)^2 r^{-4}\left(1 + \frac{v^2}{c^2}(\sin^2\theta - \cos^2\theta)\right) \tag{3}$$

and the total energy is

$$U_{EM} = \int_0^{2\pi}\int_0^{\pi}\int_{L_1}^{L_2} u_{EM}(r, \theta)\gamma^{-1}r^2\sin\theta\, dr\, d\theta\, d\phi = \left(\frac{q^2\gamma}{8\pi\epsilon_0}\right)\left(\frac{1}{L_1} - \frac{1}{L_2}\right)\left(1 + \frac{1}{3}\frac{v^2}{c^2}\right). \tag{4}$$

So, U_{EM} is not equal to $\gamma U_{0_{EM}}$. Although the whole capacitor is an isolated system, the electromagnetic fields are interacting with the capacitor shells. Let us see what happens with the transformation of the energy of the shells.

3. Energy of the shells

For an extended distribution of matter, the energy density and the stresses transform as components of a quadri-tensor of rank two [2]. In particular, for a boost in the z direction the matter density ρ in the S frame is related to the rest density ρ_0 by

$$c^2\rho = \gamma^2\left(c^2\rho_0 + \frac{v^2}{c^2}T_0^{zz}\right), \tag{5}$$

where T_0^{zz} is the component of the stress in the z direction, through a surface normal to z in the rest frame S_0. The structure of stresses in this frame can be described considering that the conducting shells are subject to electric forces that generate external tensions on the outside surface of the internal shell and on the inside surface of the external shell. The spatial sector of the stress tensor in standard spherical coordinates has the general form

$$T^{ij} = \begin{pmatrix} T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix},$$

where $T^{ij} = T^{ji}$. In the rest frame S_0, spherical symmetry implies that $T^{r\theta} = T^{r\phi} = T^{\theta r} = T^{\theta\phi} = T^{\phi r} = T^{\phi\theta} = 0$ and $T^{\theta\theta} = T^{\phi\phi}$ and that all tensor components depend only on r. So,

$$T^{ij} \equiv \begin{pmatrix} T^R(r) & 0 & 0 \\ 0 & T^L(r) & 0 \\ 0 & 0 & T^L(r) \end{pmatrix}.$$
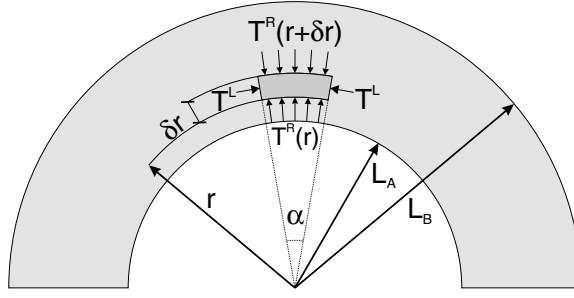
Figure 3. Forces acting on a volume element inside a shell.

The stresses are subject to the condition of equilibrium of forces in any small volume element inside a shell as shown in figure 3. The radial components of the forces acting on such a volume element are

$$F_2 = \int_0^\alpha (T^R(r)\cos\theta)(2\pi r \sin\theta)(r\,d\theta) = T^R(r)\pi r^2 \sin^2\alpha, \tag{6}$$

$$F_1 = -T_R(r + \delta r)\pi(r + \delta r)^2 \sin^2\alpha, \tag{7}$$

$$F_L = \int_r^{r+\delta r} (T^L(r')\sin\alpha)(2\pi)(r'\sin\alpha)\,dr' = 2\pi \sin^2\alpha \int_r^{r+\delta r} T^L(r')r'\,dr'. \tag{8}$$

The equilibrium condition $F_1 + F_2 + F_L = 0$ leads to

$$\frac{d}{dr}(T^R(r)r^2) = 2rT^L(r). \tag{9}$$

This equation is not sufficient for finding out the stresses inside the shells. However, as we will see, it tells us that the total energy in the moving frame depends only on the boundary values of the stresses. The Cartesian component, relevant for the total energy in equation (5), is

$$T^{zz}(r, \theta) = T^R(r)\cos^2(\theta) + T^L(r)\sin^2(\theta).$$

For each spherical shell the spatial integral of this stress, using equation (9) and integrating by parts, is

$$U_T = \int_{L_A}^{L_B} \int_0^\pi \int_0^{2\pi} \gamma^2 \frac{v^2}{c^2}(T^{zz}(r, \theta))\gamma^{-1}r^2 \sin\theta\,d\theta\,d\phi\,dr$$

$$= \gamma \frac{v^2}{c^2} \frac{4\pi}{3}((L_B)^3 T^R(L_B) - (L_A)^3 T^R(L_A)), \tag{10}$$

where L_A and L_B are the internal and external radii of the shell.

In order to calculate the tensions on the surfaces of the shells, we first have to determine the electric field acting on these surfaces. We can start considering just a single spherical surface of radius L with centre at $r = 0$ and uniform charge density $\sigma = q/4\pi L^2$. The electric field for points inside ($r < L$) or outside ($r > L$) this spherical surface can be calculated using directly the Gauss law. One finds the well-known result

$$\begin{aligned}
\vec{E}(r) &= 0 \qquad (r < L),\\
\vec{E}(r) &= \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \qquad (r > L).
\end{aligned} \tag{11}$$

For points that lie on the surface, $r = L$, we can calculate the electric field by considering an infinitesimal element of area dA and integrating the electric field contributions $d\vec{E}$ produced by all the other area elements of the surface. The result of this integration, in the limit where the area dA goes to zero, is a total electric field reduced by a factor of $1/2$ with respect to the field just outside the surface:

$$\vec{E} = \left(\frac{1}{2}\right) \frac{q}{4\pi\epsilon_0 L^2} \hat{r} \qquad (r = L). \tag{12}$$

Now, returning to our system of two shells, the external tensions act only on the surfaces where the charges are located, i.e., the external surface of the internal shell and the internal surface of the external shell. So we will consider only these surfaces. For the inside shell there is no contribution from the external shell, so the field on the charged surface is just

$$\vec{E}(L_1) = \left(\frac{1}{2}\right) \frac{q}{4\pi\epsilon_0 L_1^2} \hat{r}, \tag{13}$$

while for the external shell we have the superposition of the field produced by the internal shell (positive radial direction and no factor of $1/2$) and the field produced by the external shell (negative radial direction and factor $1/2$) resulting in

$$\vec{E}(L_2) = \left(\frac{1}{2}\right) \frac{q}{4\pi\epsilon_0 L_2^2} \hat{r}. \tag{14}$$

The radial tensions on the surfaces of the shells correspond to the inward radial components of the forces per unit area. So, for each surface, we multiply the electric field by the surface charge density and by the unitary normal pointing inside the shell. We find that the nonvanishing tensions are equal to

$$T^R(L_1) = -\frac{q^2}{32\pi^2 L_1^4 \epsilon_0}, \qquad T^R(L_2) = -\frac{q^2}{32\pi^2 L_2^4 \epsilon_0}. \tag{15}$$

So, integrating the energy density (5) in the volume of the two shells, we find contributions to equation (10) from the internal shell with $L_B = L_1$ and from the external shell with $L_A = L_2$. The energy of the shells in the S frame is thus

$$Mc^2 = \gamma M_0 c^2 - \frac{1}{3}\frac{v^2}{c^2}\frac{q^2\gamma}{8\pi\epsilon_0}\left(\frac{1}{L_1} - \frac{1}{L_2}\right). \tag{16}$$

4. Total energy

The total energy of the capacitor (shells and fields) using equations (4) and (16) is

$$U = U_{EM} + Mc^2 = \gamma(M_0 c^2 + U_{0_{EM}}) = \gamma U_0.$$

So, the total energy transforms in the expected way. Also, it does not depend on the stresses inside the capacitor shells but only on their boundary values. This fact is due to the asymmetric time delay in the electromagnetic forces acting on the different parts of the shells in the S frame when the capacitor is charged, which leads to a net work in this frame.

Acknowledgment

References

[1] Rindler W and Denur J 1988 *Am. J. Phys.* **56** 795
[2] Rindler W 1982 *Introduction to Special Relativity* (Oxford: Clarendon)