

Superluminal apparent motions in distant radio sources

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We derive the prediction of the standard model of superluminal radio sources for the apparent transverse velocity of a radio source located at redshift z . The apparent velocity of the source is reduced by a factor of $1+z$ compared to that of a similar nearby source. The cause of this reduction is recession of the distant source due to the expansion of the universe. The apparent velocity of a source can be estimated from its redshift and proper motion using the values of the Hubble constant and the mean densities of different energy components in the universe. We derive an expression for the velocity valid for the currently favored cosmological model: a flat universe with a nonzero cosmological constant. © 2005 American Association of Physics Teachers.
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I. INTRODUCTION

In 1966, Martin Rees predicted that “an object moving relativistically in suitable directions may appear to a distant observer to have a transverse velocity much greater than the velocity of light.”¹ A few years later, such motion was discovered in very distant astronomical radio sources such as radio galaxies and quasars. The motion and the sources where it took place are called superluminal, that is, faster than light. The discovery was the spectacular result of a new technique known as very long base line interferometry (VLBI). This technique has enabled the mapping of the morphologies of radio sources to accuracy better than milliseconds of arc.

Many radio galaxies and quasars contain in their nuclei compact sources of radio emission with several components that appear to move apart in successive VLBI images. Their apparent transverse velocity of separation often exceeds the speed of light. Superluminal motion has been observed in over 100 sources.² However, it is not unique to radio galaxies and quasars. There also is a class of recently discovered galactic superluminal sources called microquasars. All of these sources (galactic and extragalactic) are thought to contain a black hole, which is responsible for the ejection of mass at high velocities.

Superluminal motion does not contradict special relativity. In the generally accepted standard model, it can be explained as a light travel-time effect. Superluminal radio sources can be modeled by one or more radiating “blobs,” moving at a relativistic velocity away from a stationary “core.” Imagine a blob of matter starting at the core and moving toward an observer very fast and nearly head on. When the blob is at the core, it emits some light toward the observer. After it has moved toward the observer (and slightly to the side), it again emits light toward the observer. Because it is closer to the observer, this light will take a shorter time to travel to the observer. If we ignore this fact, then we will underestimate the true time interval, and so we will overestimate the speed.

More quantitatively, the observed transverse velocity of the separation of the blob from the core, v_a , is related to the true velocity, v , and the angle to the line of sight, θ , by

$$v_a = \frac{v \sin \theta}{1 - (v/c) \cos \theta}, \quad (1)$$

where c is the velocity of light. The angle $\theta=0$ corresponds

to motion directly toward the observer; $\theta=\pi$ corresponds to motion directly away from the observer. Equation (1) shows that if the motion is relativistic, $v \lesssim c$, and almost in the direction of the observer, separation at superluminal apparent velocities will be observed.

The derivation of Eq. (1) can be found in many textbooks on astrophysics (see, for example, Refs. 3, 4). To our knowledge, however, none of them stresses its fundamental limitation: it is valid only for nearby sources. However, most sources are actually very distant. The apparent transverse velocity of a distant radio source also depends on its redshift, which is the relative difference between the observed and the rest wavelength of the emitted light, $z \equiv (\lambda_{\text{obs}} - \lambda_{\text{em}}) / \lambda_{\text{em}}$.⁵ Specifically, the apparent transverse velocity is reduced by a factor of $1+z$.⁶ However, this effect is only briefly discussed,⁷ and its physical interpretation appears to be missing in the literature. In this paper, we derive the effect from first principles and elucidate its physical meaning.

The proper motion of a celestial object is the change in time of its angular position on the sky. To estimate the apparent velocity of an object with proper motion μ , located at the redshift z , we need its angular diameter distance, D_A . D_A depends on the redshift as well as on various cosmological parameters. To date, all analyses of superluminal radio sources at large distances employ the classical Mattig formula for D_A , which is valid only for a vanishing cosmological constant, Λ .⁹ However, cosmological observations suggest that $\Lambda \approx 0.7$ in units of the critical density.¹⁰ We will derive an expression in integral form for D_A for $\Lambda \neq 0$, corresponding to a flat universe. Following Pen,¹¹ we also obtain its analytic approximation.

The paper is organized as follows. In Sec. II we derive the apparent transverse velocity of a distant radio source in the nonrelativistic approximation. In Sec. III we present the corresponding derivation within the framework of special relativity. Finally, in Sec. IV we present a derivation based on general relativity. In Sec. V we discuss how to estimate the apparent velocity of a source from its proper motion and redshift and derive a relation for the velocity valid for the currently favored model of the universe. We conclude in Sec. VI.

II. NONRELATIVISTIC APPROXIMATION

Equation (1) is valid for a source that is at rest with respect to an observer. However, extragalactic superluminal sources

participate in the global expansion of the universe. Consequently, they have Hubble velocities of recession proportional to their distance. For distant sources these velocities are a considerable fraction of the speed of light. In this section we will investigate the effect the recession velocity of a source has on its apparent transverse velocity, in the nonrelativistic limit.

We assume that at time $t^{(1)}=0$ a blob of matter is ejected from the core of a radio source at an angle θ to the line of sight of an observer located at a distance r from the source. The observer notices this event at time $t_o^{(1)}=r/c$. At time $t^{(2)}=\Delta t$, the blob has moved to a distance $v\Delta t$ away from the source, where v is its velocity relative to the source. The transverse displacement of the blob from the core is then $\Delta y=v\sin\theta\Delta t$. If the source were stationary, this motion would be noticed by the observer at time $t_o^{(2)}=\Delta t+r/c-(v\cos\theta\Delta t)/c$, because at time $t^{(2)}$ the blob is closer to the observer by the distance $v\cos\theta\Delta t$. However, due to the expansion of the universe, the source is receding from the observer with the Hubble velocity v_H . We first assume that both v and v_H are much less than the velocity of light. Then

$$t_o^{(2)}=\Delta t+\frac{r+\Delta r}{c}-\frac{v\cos\theta\Delta t}{c}, \quad (2)$$

where $\Delta r=v_H\Delta t$. In Eq. (2), we have implicitly used the Galilean transformation of velocities, that is, the blob velocity relative to the observer is a sum of the source velocity relative to the observer and the blob velocity relative to the source. Hence, the relation between the observed and true time intervals, respectively Δt_o and Δt , from the moment of emanation of the blob from the core, is

$$\Delta t_o \equiv t_o^{(2)} - t_o^{(1)} = (1 + \beta_H - \beta \cos \theta) \Delta t, \quad (3)$$

where $\beta_H = v_H/c$, and the apparent transverse velocity of the blob measured by the observer is

$$v_a = \frac{\Delta y}{\Delta t_o} = \frac{v \sin \theta}{1 + \beta_H - \beta \cos \theta}. \quad (4)$$

For $\beta_H=0$, Eq. (4) reduces to Eq. (1), which is valid for a blob ejected from a *stationary* source.

From Eq. (4) it is clear that the recession of the source (away from the observer) opposes the effect of the blob motion (toward the observer), in a sense that it lowers the value of the apparent transverse velocity of the blob. For nonrelativistic recession we have $v_H = cz$, where z is the redshift of the source. This equation yields $\beta_H = z$, and hence

$$\beta_a = \frac{\beta \sin \theta}{1 + z - \beta \cos \theta}. \quad (5)$$

We reiterate that Eq. (5) is valid only when both z and β are much less than unity.

III. SPECIAL-RELATIVISTIC APPROACH

To appear superluminal, the velocity of a blob must be relativistic. In this section we apply special relativity to derive a formula for the apparent velocity of a blob, moving away from a core located at a redshift z . According to special relativity, the Cartesian components of velocities measured in the reference frames O and O' , moving with constant relative velocity V along the x axis, are related by

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2}, \quad (6a)$$

$$v_y = \frac{v'_y}{\gamma(1 + v'_x V/c^2)}, \quad (6b)$$

where $\gamma = [1 - (V/c)^2]^{-1/2}$. Unprimed (primed) quantities refer to the velocity components measured in the frame $O(O')$. We align the x axis along the observer's line of sight to the radio source. The unprimed frame is the observer's reference frame; the primed one is the source's frame. Then, $v'_x = -v' \cos \theta$ and $v'_y = v' \sin \theta$, where v' is the blob velocity relative to the source. The source velocity relative to the observer is $V = v_H$. Hence, the Cartesian components of the blob velocity relative to the observer are

$$v_x = \frac{v_H - v' \cos \theta}{1 - \beta' \beta_H \cos \theta}, \quad (7a)$$

$$v_y = \frac{v' \sin \theta}{\gamma_H(1 - \beta' \beta_H \cos \theta)}, \quad (7b)$$

where $\gamma_H = (1 - \beta_H^2)^{-1/2}$.

In Eq. (7) the time dilation effect is naturally accounted for. However, in the observer's inertial frame we have to take into account the extra time dilation factor that occurs because the distance to the emitting blob (and thus the distance light has to propagate to reach the observer) is changing. In the time Δt the emitter moves a distance $v_x \Delta t$ away from the observer (v_x and the distance may be negative). The total observed time, Δt_o , is Δt plus the extra factor describing how long it takes light to traverse this extra distance ($v_x \Delta t/c$)

$$\Delta t_o = \Delta t + \frac{v_x \Delta t}{c} = (1 + \beta_x) \Delta t. \quad (8)$$

Note that Eq. (8) is similar to Eq. (3), but instead of the nonrelativistic approximation for v_x , $v_x = v_H - v' \cos \theta$, we have used the special relativity formula (7a). The transverse distance covered by the blob in the time Δt is $\Delta y = v_y \Delta t$. Hence, the apparent transverse velocity is $v_a = \Delta y / \Delta t_o = v_y / (1 + \beta_x)$. From Eq. (7) we have

$$v_a = \frac{v' \sin \theta}{\gamma_H(1 - \beta' \beta_H \cos \theta)} \left(1 + \frac{\beta_H - \beta' \cos \theta}{1 - \beta' \beta_H \cos \theta} \right)^{-1} \quad (9)$$

or

$$\beta_a = \frac{\gamma_H^{-1} \beta' \sin \theta}{(1 + \beta_H)(1 - \beta' \cos \theta)}. \quad (10)$$

Next, we have

$$\frac{\gamma_H^{-1}}{1 + \beta_H} = \frac{(1 - \beta_H^2)^{1/2}}{1 + \beta_H} = \left(\frac{1 - \beta_H}{1 + \beta_H} \right)^{1/2} = (1 + z)^{-1}. \quad (11)$$

In the last step we have used the expression for the special relativistic Doppler effect. The result is

$$\beta_a = \frac{\beta \sin \theta}{(1 + z)(1 - \beta \cos \theta)}. \quad (12)$$

For consistency of notation, we have omitted the primes on β in Eq. (12). However, it should be clear that in Eq. (12), as in Eq. (5), $v = \beta c$ is the blob velocity *relative to the source*.

Equation (12) shows that the recession (Hubble) velocity of a radio source reduces the amplitude of the apparent transverse motion of its blob by the factor of $1+z$. When both z and β are much less than unity, the term $z\beta \cos \theta$ can be neglected and Eq. (12) reduces to its nonrelativistic limit, Eq. (5). What is the range of applicability of Eq. (12)? In its derivation we have not made any explicit assumptions about the values of z and β , except that $z \geq 0$ and $\beta \in [0,1]$. However, when applied to the real world, special relativity is guaranteed to work only locally, because, in general, it is not possible to eliminate the gravitational field globally by a suitable choice of a reference frame. (There are no global inertial frames in the universe.) Thus, special relativity applies only to a limited region around a radio source. Consequently, though it has not been assumed explicitly, Eq. (12) is valid for arbitrary $\beta \in [0,1]$, but only for $z \ll 1$. To find its generalization for arbitrary z , we need to apply general relativity.

IV. GENERAL-RELATIVISTIC APPROACH

The metric of a homogeneous and isotropic universe is given by the Robertson-Walker line element

$$c^2 ds^2 = c^2 dt^2 - a^2(t) [d\chi^2 + R_0^2 S^2(\chi/R_0) (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (13)$$

Here, R_0^{-2} is the curvature of the universe and the function $S(x)$ equals $\sin(x)$, x , and $\sinh(x)$ for a closed, flat, and open universe, respectively. The function $a(t)$ is called a scale factor and relates the physical, or proper, coordinates of a galaxy, $\mathbf{r}$, to its fixed or comoving coordinates, χ : $\mathbf{r} = a\chi$. This function accounts for the expansion of the universe; its detailed time dependence is determined by the Friedman equations.¹² We normalize a so that at the present time, $a(t_0) = 1$.

Photons propagate along null geodesics, $ds=0$. If we place an observer at the origin of the coordinate system, the geodesic of the photons emitted by the source toward the observer is radial. The source's comoving radial coordinate is χ . From the metric (13), we have for the photons emitted from the core

$$\int_{t_e^{(1)}}^{t_o^{(1)}} \frac{cdt'}{a(t')} = \int_0^\chi d\chi' = \chi. \quad (14)$$

For the photons emitted later from the blob

$$\int_{t_e^{(2)}}^{t_o^{(2)}} \frac{cdt'}{a(t')} = \chi - \Delta\chi, \quad (15)$$

where $\Delta\chi$ is the comoving distance the blob has covered, projected on the line of sight. Its relation to the proper distance, $\Delta\ell$, is $\Delta\ell = a(t_e)\Delta\chi = (1+z)^{-1}\Delta\chi$, where z is the source's redshift. The proper distance is $v \cos \theta \Delta t_e$. Hence

$$\Delta\chi = (1+z)v \cos \theta \Delta t_e. \quad (16)$$

The subtraction of Eq. (15) from Eq. (14) yields

$$\Delta\chi = \left[\int_{t_e^{(1)}}^{t_o^{(1)}} - \int_{t_e^{(2)}}^{t_o^{(2)}} \right] \frac{cdt'}{a(t')} = \int_{t_e^{(2)}}^{t_e^{(1)}} \frac{cdt'}{a(t')} - \int_{t_o^{(1)}}^{t_o^{(2)}} \frac{cdt'}{a(t')}. \quad (17)$$

Because both Δt_e and Δt_o are very small compared to the Hubble time, Eq. (17) simplifies to

$$\Delta\chi + \frac{c\Delta t_o}{a(t_o)} = \frac{c\Delta t_e}{a(t_e)} \quad (18)$$

or

$$\Delta\chi + c\Delta t_o = (1+z)c\Delta t_e. \quad (19)$$

By using Eq. (16), we obtain

$$\Delta t_o = (1+z)\Delta t_e(1 - \beta \cos \theta). \quad (20)$$

For vanishing blob velocity, Eq. (20) describes the well-known phenomenon of cosmological time dilation. The transverse component of the distance covered by the blob is $\Delta y = v \sin \theta \Delta t_e$. The apparent transverse velocity is $\Delta y / \Delta t_o$, and hence

$$\beta_a = \frac{\beta \sin \theta}{(1+z)(1 - \beta \cos \theta)}, \quad (21)$$

in agreement with Eq. (12). Thus, in this case, the prediction of special relativity turns out to be valid globally, that is, for arbitrary z . Why?

The amplitude of the apparent transverse motion of a distant radio source is reduced by a factor that depends only on its redshift; it does not depend on the background cosmological model. In particular, it does not depend on the mean densities of the different energy components in the universe, Ω_Λ and Ω_m . Here, Ω_Λ denotes the cosmological constant Λ expressed in units of the critical density, and Ω_m is the mean density of nonrelativistic matter in the universe, also in units of the critical density. Physically, these components cause the universe's acceleration and deceleration, respectively. Therefore, the lack of sensitivity of the reduction factor (of the apparent transverse velocity) to their densities implies that the factor is insensitive to the acceleration or deceleration of the cosmological expansion, and hence its origin is kinematic. Mathematically, the reduction is the same for any Ω_Λ and Ω_m , so in particular for $\Omega_\Lambda = \Omega_m = 0$, that is, for an empty universe. This particular cosmological model is called the Milne model, or *kinematic cosmology*. The dynamics of an empty universe can be completely described by special relativity,¹³ which is why its prediction turns out to be valid globally here. However, this conclusion holds only *a posteriori*, that is, after applying general relativity and finding out that the reduction factor depends only on the redshift of the source.

Quantitatively, the amplitude of the apparent transverse motion of a radio source located at a redshift z is reduced by a factor $1+z$. The general relativistic explanation of this fact is the cosmological time dilation between the observer's frame and the frame of the object.⁶ The special relativistic explanation is the recession velocity of the object, resulting in the same amount of time dilatation. These explanations are mutually consistent, because the origin of cosmological time dilation is the expansion of the universe, which causes distant galaxies to recede from the Milky Way. We have argued that the reduction of the apparent velocity is in essence a kinematic effect and can be qualitatively explained within a non-relativistic framework, as demonstrated in Sec. II. Relativistic corrections are necessary to describe the effect quantitatively.

V. ESTIMATING THE APPARENT VELOCITY

The physical size of an object at the redshift z that subtends the angle $\Delta\phi$ on the sky, Δy , can be readily derived from the metric in Eq. (13). The result is

$$\Delta y = D_A \Delta\phi. \quad (22)$$

Hence, the apparent transverse velocity of a radio source is

$$v_a = \mu D_A, \quad (23)$$

where $\mu = \Delta\phi/\Delta t_o$ is its observed proper motion. By measuring the redshift and the proper motion of a radio source and knowing the cosmological parameters, we can estimate its apparent velocity. This estimate can be subsequently used to constrain the combination of the parameters characterizing the source, given by the right-hand side of Eq. (21). To constrain only the internal parameters of the source that describe its kinematics and geometry (that is, v and θ), it is necessary to eliminate the dependence of v_a on the redshift. Therefore, instead of v_a itself, it is common to estimate the quantity

$$v_m = (1+z)v_a. \quad (24)$$

Unfortunately, v_m is widely called the “apparent velocity” (see, for example, Refs. 6, 14–17). This term is misleading, because it disguises the fact that the true apparent velocity, v_a , is affected by the recession velocity of the source. We will therefore distinguish between v_a and v_m , and will call the latter the “velocity measure.” The latter is the velocity that an observer would measure if he/she were located at the redshift z in the vicinity of the source. This velocity is not the velocity we measure on Earth. We have

$$v_m = (1+z)\mu D_A = \mu D, \quad (25)$$

where $D = (1+z)D_A$ is the distance measure. We recall that an object of luminosity L has flux $f = L/(4\pi D_L^2)$, where

$$D_L = (1+z)D = (1+z)^2 D_A \quad (26)$$

is the luminosity distance.

To date, all analyses of superluminal motion in extragalactic radio sources have assumed that the universe has a vanishing cosmological constant at low redshifts dominated by nonrelativistic matter.¹⁸ For such a universe, Mattig¹⁹ derived an analytical expression for the distance measure D for arbitrary Ω_m . Consequently, the classical Mattig formula for D has been used in all analyses of superluminal radio sources (see, for example, Refs. 6, 14–16, 20). However, observations in cosmology consistently imply that Λ is about 0.7 in units of the critical density (that is, $\Omega_\Lambda \approx 0.7$). The distance measure D (or the luminosity distance D_L) is sensitive to the presence of Λ . Thanks to this sensitivity, the Hubble diagram for supernovae Ia has been successfully used to show that the universe has a nonzero Λ . Therefore, we need a relation for the distance that accounts for the presence of Λ .

Observations also strongly suggest that the universe must be very close to spatially flat (see, for example, Ref. 21). For a flat universe with a cosmological constant, the luminosity distance is (see, for example, Ref. 22)

$$D_L(z) = cH_0^{-1}(1+z) \int_0^z [1 - \Omega_m + \Omega_m(1+x)^3]^{-1/2} dx. \quad (27)$$

Here, H_0 denotes the Hubble constant, and spatial flatness requires $\Omega_\Lambda = 1 - \Omega_m$. For small redshifts, $z \ll 1$, Eq. (27) reduces to $D_L = cH_0^{-1}z$. The integral in Eq. (27) cannot be performed analytically; it is a Weierstrass elliptic function.²³

Pen¹¹ derived an approximate analytic expression for $D_L(z)$, accurate to better than 0.4% for $0.2 \leq \Omega_m \leq 1$, for any redshift. The value of Ω_m is currently known to be around 0.3. Pen's approximation for the luminosity distance is

$$D_L = cH_0^{-1}(1+z) [\eta(0, \Omega_m) - \eta(z, \Omega_m)], \quad (28)$$

where

$$\begin{aligned} \eta(z, \Omega_m) = & 2\Omega_m^{-1/2} [(1+z)^4 - 0.1540(1+z)^3 s \\ & + 0.4304(1+z)^2 s^2 + 0.19097(1+z)s^3 \\ & + 0.066941s^4]^{-1/8} \end{aligned} \quad (29)$$

and

$$s = \left(\frac{1 - \Omega_m}{\Omega_m} \right)^{1/3}. \quad (30)$$

(The original notation has been slightly modified.) It is a matter of choice whether to use a simple numerical integral or its fairly complex analytic approximation. If one chooses the latter approach, then from Eqs. (25) and (28), the velocity measure v_m , or $\beta_m = v_m/c$, is

$$\beta_m = \mu H_0^{-1} [\eta(0, \Omega_m) - \eta(z, \Omega_m)], \quad (31)$$

with η given by Eq. (29). Note that β_m is dimensionless, as it should be. Equation (24) implies that

$$\beta_a = \mu H_0^{-1}(1+z)^{-1} [\eta(0, \Omega_m) - \eta(z, \Omega_m)]. \quad (32)$$

For completeness, we provide current estimates of the Hubble constant and Ω_m . From a joint analysis of the SDSS and the Wilkinson microwave anisotropy probe (WMAP) data, Tegmark *et al.*²⁴ deduced that $H_0 = 70_{-3}^{+4} \text{ km s}^{-1} \text{ Mpc}^{-1}$, and $\Omega_m = 0.30 \pm 0.04$ (68% confidence interval).

VI. CONCLUSIONS

The apparent transverse velocity of a radio source located at a redshift z is suppressed by a factor $1+z$. We have derived this result within the framework of both special and general relativity. The underlying cause of this suppression is the recession of the source due to the expansion of the universe.

Given the values of the Hubble constant and the mean densities of different energy components in the universe, the apparent velocity of a source can be estimated from its redshift and proper motion. We have derived a relation for the velocity valid for the currently favored cosmological model, that is, a flat universe with a nonzero cosmological constant.

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³J. H. Krolik, *Active Galactic Nuclei: From the Central Black Hole to the Galactic Environment* (Princeton University Press, Princeton, 1999), p. 296.

⁴F. H. Shu, *The Physical Universe* (University Science, Mill Valley, CA, 1982), p. 313.

⁵In other words, the redshift is a shift in the frequency of photons toward lower energy, or longer wavelength, that is, the red part of the electromagnetic spectrum. The more distant is the source, the higher is its redshift.

⁶T. J. Pearson and J. A. Zensus, "Superluminal radio sources: Introduction," in *Superluminal Radio Sources*, edited by J. A. Zensus and T. J. Pearson (Cambridge U.P., Cambridge, 1987), pp. 1–12.

⁷"To derive the apparent transverse velocity β_{app} from an observed proper motion μ it is necessary to take account of the time dilation between the observer's frame and the frame of the object: $\beta_{\text{app}} = \mu D_A (1+z)/c$ " (see Ref. 6). D_A is the angular diameter distance and the extra factor of $1+z$ is due to time dilation. If we combine this relation with the relation between the angular diameter distance and the luminosity distance, Eq. (26), and a nonstandard form of the Mattig formula for the latter (see Ref. 8), we obtain an expression for the apparent transverse velocity [Eq. (4) of Ref. 6], repeatedly used in the literature.

⁸J. A. Peacock, *Cosmological Physics* (Cambridge U.P., Cambridge, 1999), Eq. (3.79).

⁹The cosmological constant, Λ , occurs in Einstein's theory of general relativity. It is proportional to the energy density of the vacuum. The constant can be thought of as the amount of energy that is embedded in empty space. If it is positive, then the expansion of space would release more energy which in turn would accelerate the expansion; if it is negative, the expansion of space would consume energy. Recent observations indicate that the expansion of the universe is accelerating and if this acceleration is due entirely to the cosmological constant, then its value is on the order of 10^{-35} s^{-2} , or about 0.7 in units of the critical density (see Ref. 10).

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¹⁸An exception is Ref. 2, where it was assumed that $\Omega_\Lambda = 0.7$. However, no details are given of the calculation of the apparent velocity from the proper motion.

¹⁹W. Mattig, "Über den Zusammenhang zwischen Rotverschiebung und scheinbarer Helligkeit," *Astron. Nachr.* **284**, 109–111 (1958).

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