A circular twin paradox

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In the special relativistic twin paradox presented here, each twin lives on one ring of a counterrotating pair of infinitesimally separated rings, so that the twins travel on the same circular path but in opposite directions. The observers on the ring of one twin should see the clock of the other twin slowed by time dilation, but at each meeting of the twins symmetry demands that they agree on the amount of time that has passed since their previous meeting. The resolution of the paradox focuses attention on the relation of time dilation to clock synchronization.

I. INTRODUCTION

Twin paradoxes have played an important role in the pedagogical history of special relativity theory; generations of physics students have been challenged by their professors to apply their newly minted understanding of relativistic principles to the resolution of these famously counterintuitive problems. Continuing in this venerable tradition, we present a twin paradox here, but one that differs significantly from the familiar rocket parable. Although this paradox appears in Lightman *et al.*,[1] the solution provided there is brief, mathematical, and formal, and is not accessible to the full range of students for whom the paradox would be of interest. Some elements of the resolution of the paradox are related to issues that have been discussed by several authors[2–4] but the connection to our paradox is not straightforward. Here we present an analysis that should be both mathematically and physically intelligible to beginning relativity students. In order to optimize the accessibility of the basic ideas, we relegate all calculations to the Appendix.

In our parable, Lisa, the protagonist of the paradox, lives on a ring of radius R with a team of observers stationed at every point on her ring. Bart lives on an identical ring. The rings are negligibly separated, rotating at equal and opposite angular velocity ω about a common axis. This common axis is at rest in a ''Lab'' frame, through which Bart, Lisa, and all of Lisa's observers move at speed $v=\omega R$. Bart moves through the Lab at this speed in the counterclockwise direction while the observers on Lisa's ring move through the Lab at this speed in the clockwise direction. The twins will pass each other periodically, their negligible separation allowing each to read the other's clock.

At a certain moment, Bart and Lisa happen to be at the same place, and they notice that their clocks both read $t=0$. To the observers on Lisa's ring (see Fig. 1) Bart's clock flies by at speed $v_{\rm rel}=2v/(1+v^2/c^2)$. (Recall that relative speeds do not add simply in special relativity.) Due to time dilation, Lisa's team observes Bart's clock ticking more slowly than their own clocks by a relative Lorentz factor $\gamma_{\rm rel}$ that works out to be $\gamma_{\rm rel}=(1+v^2/c^2)/(1-v^2/c^2)$. Since Bart's clock agrees with Lisa's as he passes her at $t=0$, time dilation means that his clock will lag behind the clock of the next of Lisa's observers that he passes. As he passes successive members of Lisa's observing team, his clock should be seen to lag further and further behind. One half-rotation later, the observer he passes will be Lisa. We conclude that when Bart passes her their clocks will disagree; Bart's clock must lag behind Lisa's.

This is nonsense, of course. Very convincing arguments support our intuition that Bart's and Lisa's clocks will agree at each meeting if they agree at the first meeting. One of these arguments follows from the point of view of the laboratory observers. They see Bart and Lisa tracing out identical motions, except one is going clockwise and the other counterclockwise. This ''handedness'' of the motion can have no effect on the rate of ticking of clocks, so the clock of Bart and the clock of Lisa must tick off the same number of seconds between meetings (though neither ticks off the same number of seconds as the Lab clocks). Another obvious ''twin'' argument further convinces us that Bart's clock cannot lag Lisa's at their subsequent meetings: In the very same way we argued that Bart's clock must lag behind Lisa's we could have argued (based on a corps of special relativity observers on Bart's ring) that Lisa's clock will lag behind Bart's at future meetings.

''Clearly'' there is something wrong with our argument about time dilation and Bart's clock, but what precisely is wrong?

II. CLOCK SYNCHRONIZATION ON A ROTATING RING

Special relativistic time dilation must be considered in the context of clock synchronization. Suppose that Lisa's closest neighbor in the counterclockwise direction is Milhouse. As Bart passes first Lisa and then Milhouse, time dilation is inferred from the comparison of Bart's clock with Lisa's and then with Milhouse's. We have set up our parable so that the clocks of Bart and Lisa both read $t=0$ at the first event. The comparison at the second event (to determine whether Bart's clock lags, leads, or neither) depends on the setting of Milhouse's clock, which in turn depends on how *his* clock was originally related to Lisa's.

Time dilation will only be observed from a reference frame in which the clocks are appropriately synchronized. Clarification of our paradox thus requires careful deconstruction of the clock synchronization issues implicit in the situation we have described.[5] One prescription for synchronizing two clocks is that given by Einstein and called ''Einstein synchronization'' in the special relativity literature. In this prescription for clocks A and B,[6] ''...a common time for A and B...cannot be defined at all unless we establish by definition that the 'time' required by light to travel from A to B

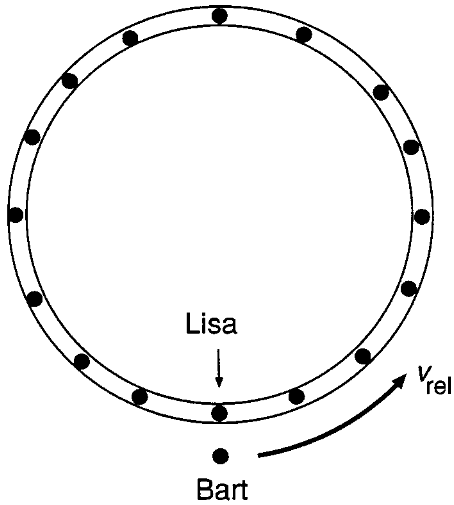

Fig. 1. As viewed in a rotating frame in which Lisa's ring is unmoving, Bart moves in the counterclockwise direction, past the observers on Lisa's ring, at a speed $v_{\rm rel} = 2v/(1+v^2/c^2)$.

equals the 'time' it requires to travel from B to A. Let a ray of light start [as shown in Fig. 2] at the A time t_A from A towards B. At the 'B time' t_B let it be reflected from B in the direction of A, and arrive again at A in the 'A time' t'_A. The two clocks are Einstein synchronized if

$$t_B - t_A = t'_A - t_B.\text{''} \tag{1}$$

The synchronization of clocks on *rotating* rings presents a special challenge. Here we describe four ways one might choose to synchronize such clocks. In the first two methods, the rings are imagined to be initially nonrotating, that is, initially at rest in the Lab. The clocks are synchronized while at rest, and then the rings are set into rotation. In the last two methods, synchronization is done while the rings are already rotating.

A. Method 1

Before the rings are set into motion, the ring observers, sitting at rest in the Lab, may decide to synchronize their clocks according to the principles of Einstein synchronization, i.e., by exchanging light signals. Lisa, ensconced at point A, noting that her clock registers t_A, fires a laser beam at Milhouse, her ''next door neighbor'' in the counterclockwise direction, who is stationed at point B. At t_B, he receives and reflects the beam back to her; she receives the signal at t'_A. Lisa sends Milhouse a slip of paper upon which is written the value of $(t'_A + t_A)/2$, with instructions that his clock should have had that reading at t_B. Milhouse adjusts his clock accordingly. This procedure is followed from observer to observer around the ring, and we imagine the limit of an infinite number of observers with infinitesimal separation. Einstein synchronization on the not-yet spinning rings thus allows Lisa's team to constitute themselves correctly as a special relativity reference frame; that is, Lisa's observers will measure the rate of a clock moving by them at speed v to be too slow by $1/\sqrt{1-v^2/c^2}$.

With clocks synchronized in what appears to be an incontrovertibly correct way, the rings are now uniformly set into rotation. (Here ''uniformly'' means that all points on the ring are treated identically.)

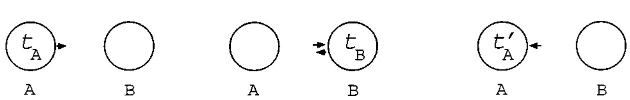

Fig. 2. Einstein synchronization of clock B with clock A. A light signal is sent from A to B and back to A. Clock B is adjusted so that $t_B = (t_A + t'_A)/2$.

B. Method 2

Before the rings are set into uniform motion, that is, when they are at rest in the Lab frame, the clocks on Lisa's ring can be synchronized by an even easier method. Upon receiving a flash from a ''Big Lab Clock''[2] stationed at the center of rotation, the ring observers can all set their clocks to read $t=0$. It seems obvious (and is true!) that this method produces the same results as the classical Einstein synchronization used in the first stage of Method 1. As in Method 1, the rings are gradually and uniformly put into motion after the synchronization process is completed.

C. Method 3

In this method, Lisa's crew chooses to synchronize their clocks with their ring already in motion. A rather obvious way to do this is to have a light flash at the center of rotation, all observers having been instructed to set their clocks to $t=0$ at the moment they see the flash. This method is the same as Method 2 except that the rings are in motion when the procedure is performed.

D. Method 4

In Method 4, Lisa and her team of observers attempt to employ Einstein synchronization on the already-rotating ring. Lisa sends a light signal to Milhouse, the first observer in the counterclockwise direction, and an infinitesimal distance from her. That observer sends back a light signal. The times of arrival of the light signals are used in the same manner as in Method 1 to synchronize the clocks. This synchronization is then continued, proceeding around the ring in the counterclockwise direction.

E. Comparison

In Method 1, before the rings are put into rotation, the clocks will be correctly synchronized for Lisa's crew to make standard special relativity observations. If the Lab observers also have clocks that are correctly synchronized, then simultaneity will mean the same thing to Lisa's observers and to the Lab observers. When the clocks of each of Lisa's observers strike midnight, the clock of each nearby Lab observer will have the same reading—say 2:23 a.m.

It is clear that this is also true for Method 2. Since the ''flash at the center'' process favors no particular location on the ring, a moment of simultaneity on the ring (the same clock reading for all of Lisa's observers) is also a moment of simultaneity in the Lab (the same reading for all Lab observer clocks).

This very same argument applies equally well to the synchronization by Method 3. Though the ring is now moving during the synchronization process, the ''flash at the center'' again favors no particular observer. It follows that a moment of simultaneity (all clocks have the same reading) on a ring will also be a moment of simultaneity in the Lab.

From the above arguments we conclude that Methods 1, 2, and 3 all provide the same sort of synchronization. As we will show in the following, Method 4 is different.

III. THE PARADOX RESOLVED

If Lisa and her nearest neighbor Milhouse were properly synchronized to be part of a special relativity reference frame moving through the Lab, then events (like the striking of midnight) that are simultaneous to Lisa and Milhouse cannot be simultaneous in the Lab frame. If Methods 1, 2, or 3 are used for synchronization of ring clocks, then events that are simultaneous to Lisa and Milhouse will also be simultaneous to the Lab observers. It follows that Lisa and Milhouse, and more generally the entire set of observers on Lisa's ring, are not correctly synchronized to constitute special relativity reference frames. This explains what we already know must be true: There will be no lagging of Bart's clock observed as it passes each of Lisa's observers. *For the relativistically inappropriately synchronized clocks of Lisa's observers, there is no time dilation of Bart's clock.*

This conclusion may seem to some readers to evade, not resolve, the paradox. We have explained away the awkward implications of time dilation by using synchronization that does not produce time dilation. We justify the inclusion of these ''inappropriate'' methods of synchronization with two arguments: First, this helps to emphasize the connection between clock synchronization and time dilation; second, a student would ask why such obvious methods of synchronization are not used.

In any case, the paradox cannot be evaded if Lisa's clocks are synchronized by Method 4. In this case, the clocks of Lisa and Milhouse are ''correctly'' synchronized. If Lisa and Milhouse are negligibly separated on the ring, their readings will differ negligibly from readings done in a standard special relativity reference frame that is instantaneously comoving with them. Lisa and Milhouse *will* therefore observe Bart's clock to run slowly and Bart's clock *will* lag Milhouse's when they pass each other. The resolution of the paradox now takes a very different form: When Method 4 is used, there will be a discontinuity in synchronization. Suppose Einstein synchronization is used starting with Lisa, proceeding to Milhouse, and proceeding around the ring until the clock of the last observer, call her Selma, is synchronized. In the case that the angular separation of Lisa's team of observers is negligibly small, Lisa at angle 0 and Selma at angle 2π are at the same place. But their clocks will not agree. Due to the discontinuity of Method 4 synchronization, Selma's clock will lead (i.e., will have a higher reading) than Lisa's by an amount that we show in the Appendix to be $Disc=(2\pi R/c)(v/c)/\sqrt{1-v^2/c^2}$.

When calculated in detail, the time dilation of Bart's clock, moving past Lisa's observers, turns out to show that Bart's clock will lag Selma's by precisely the amount *Disc*, when Bart reaches Selma. At the very same location as Selma is Lisa, whose clock lags Selma's by *Disc*, and hence agrees perfectly with Bart's clock.

The Appendix gives the details of computation of the discontinuity *Disc*, and shows that this discontinuity makes time dilation compatible with the comparison of Lisa's clock to Bart's and to the clock of a Lab observer.

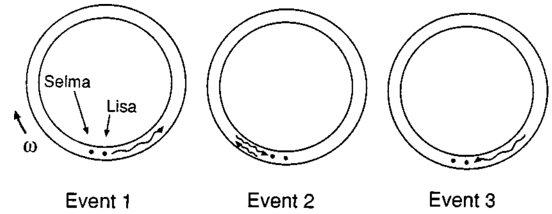

Fig. 3. Method 4 used to synchronize Lisa's clock with that of another observer.

IV. CONCLUSION

The resolution of the paradox of the counterrotating twins depends on the method that is used to synchronize clocks on a ring. If the clocks are synchronized ''uniformly'' (with no particular position on the ring singled out), then the resolution is that the observers so synchronized will not measure any time dilation. If, on the other hand, the clocks are synchronized by Einstein synchronization, starting with one particular observer, there will be a discontinuity in synchronization at the location of that observer, and this discontinuity permits both time dilation and the agreement of the twins' clocks at every meeting.

APPENDIX: DISCONTINUITY DUE TO METHOD 4 SYNCHRONIZATION

Figure 3 shows the three events needed to synchronize the clock of another observer with Lisa's clock. Let us say that Lisa is at Lab position $\phi=0$, and she will synchronize clocks with one of her observer friends at Lab angle $\phi=\Delta\phi$. (In Fig. 3, the friend is shown as Selma, the last of Lisa's observers, located at $\phi=\Delta\phi=2\pi$.) In the calculations below, the coordinate ϕ is measured relative to the Lab system. The measure $\Delta\phi$ refers to the angular position of the observer friend as measured on Lisa's ring, but it has the same value as observed in the Lab frame. In either frame it simply indicates the angular displacement of the observer as a fraction of a complete circle. If the observer friend were diametrically opposite Lisa, both the observers on Lisa's ring and the Lab observers would describe the angular difference as $\Delta\phi=\pi$.

For definitiveness let us say that Lisa's ring of observers is rotating through the Lab in the clockwise, or negative, direction, at angular velocity ω, as shown in the figure. This means that Bart will be traveling in the counterclockwise, or positive, direction relative to Lisa's observers; if Bart is to be observed by a synchronized team of Lisa's observers, synchronization must proceed in the counterclockwise direction.

Event 1 of the synchronization procedure is for Lisa to send a photon in the positive direction. Let us say that this event occurs at Lab time $t=0$. At event 2, Lisa's friend receives that photon and sends one back to Lisa in the negative direction. Between events 1 and 2 the photon moves through the Lab according to $\phi=ct/R$, and the position of Lisa's friend is given by $\phi=\Delta\phi-\omega t$. By solving these two equations for the intersection of friend and photon, we find that event 2 occurs at Lab angular location and at Lab time

$$\phi_2=\frac{\Delta\phi}{1+v/c}, \quad t_2=\frac{R}{c}\frac{\Delta\phi}{1+v/c}, \tag{2}$$

where $v\equiv\omega R$ is the speed of Lisa's observers through the Lab. Between events 2 and 3, Lisa moves through the Lab according to $\phi=\omega t$, and the photon from her friend moves through the Lab according to $\phi=\phi_2-c(t-t_2)/R$. These two motions intersect at event 3, at Lab time:

$$t_3=2\frac{R}{c}\frac{\Delta\phi}{1-v^2/c^2}. \tag{3}$$

Due to the time dilation of Lisa's clock with respect to the Lab system, Lisa's clock will read $\tau_3=t_3/\gamma$ at event 3, where $\gamma\equiv 1/\sqrt{1-v^2/c^2}$, or $\tau_3=2(R/c)\gamma$. According to the prescription for Einstein synchronization, Lisa's friend, at $\phi=\Delta\phi$, will be given instructions to adjust her clock so that it would have read

$$\text{clock setting at event } 2=(\tau_3+\tau_1)/2=\tau_3/2=(R/c)\gamma\Delta\phi \tag{4}$$

at event 2.

Now let us suppose that Lisa's friend is Selma, at $\phi=2\pi$, so that she is at the same position as Lisa. At event 2, the Lab time, from Eq. (2), is $t_2=(R/c)\Delta\phi/(1+v/c)$, and Lisa's clock reads t_2/γ, while Selma's Einstein synchronized clock reads $\tau_3/2=(R/c)\gamma\Delta\phi$. Thus at event 2 Lisa's clock will lag Selma's clock by $\tau_3/2-t_2/\gamma$, and hence by

$$Disc=\frac{\tau_3}{2}-\frac{t_2}{\gamma}=\frac{2\pi Rv}{c^2}\gamma, \tag{5}$$

where we have used $\Delta\phi=2\pi$.

1. Bart's time dilation lag

Consider Bart moving in the counterclockwise direction past Lisa's observers, with his relative velocity $v_{\text{rel}}=2v/(1+v^2/c^2)$ and relative Lorentz factor $\gamma_{\text{rel}}=(1+v^2/c^2)/(1-v^2/c^2)$. The rate at which Bart's clock advances, compared to the readings on the clocks he passes, is given by the usual time dilation relationship $\Delta t_{\text{Bart}}=\Delta t_{\text{LR}}/\gamma_{\text{rel}}$. Here the subscript "LR" indicates "Lisa's ring." Bart is comparing clocks not with Lisa, but with other observers on her ring. At the completion of his circumnavigation of Lisa's ring, he will encounter Selma, and his clock will read $\tau_{\text{Selma}}/\gamma_{\text{rel}}$, and hence will lag behind hers by $\tau_{\text{Selma}}(1-1/\gamma_{\text{rel}})$. To find Selma's clock reading at that event we note that Lisa has moved through the Lab by π and therefore Lisa's clock will read $\pi/(\omega\gamma)$. Selma's must therefore read

$$\tau_{\text{Selma}}=\pi/(\omega\gamma)+Disc=\frac{\pi\gamma}{\omega}\left(1+\frac{v^2}{c^2}\right). \tag{6}$$

Bart's reading will lag by

$$\tau_{\text{Selma}}\left(1-\frac{1}{\gamma_{\text{rel}}}\right)=\frac{\pi\gamma}{\omega}\left(1+\frac{v^2}{c^2}\right)\left(1-\frac{1}{\gamma_{\text{rel}}}\right)=2\pi\gamma Rv/c^2. \tag{7}$$

But this is the same as the amount *Disc*, by which Lisa's clock lags behind Selma's! Thus Bart's clock *does* undergo time dilation. It lags behind Selma's clock. But due to the discontinuity in synchronization, Lisa's clock lags by precisely the same amount, and Bart and Lisa will have clock readings that agree, as of course they must.

2. Time dilation of a Lab observer

Consider now a Lab observer, that is, an observer fixed in position in the Lab frame. Suppose that such an observer happens to be fixed at an infinitesimal distance from Lisa's ring. Such an observer will be measured by Lisa's observers to be moving past them, as shown in Fig. 4, at speed v in the counterclockwise direction. While a study of clock comparisons between this observer and observers on Lisa's ring is not necessary to resolve the Bart–Lisa paradox, it does further illuminate the role of the synchronization discontinuity on Lisa's ring.

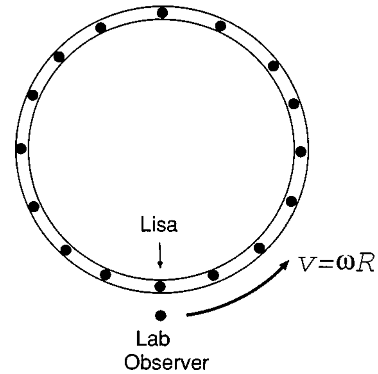

Fig. 4. As viewed in a rotating frame in which Lisa's ring is unmoving, an observer who is fixed in the Lab frame moves in the counterclockwise direction, past the observers on Lisa's ring, at a speed $v=\omega R$.

Let us suppose that at the moment the Lab observer passes Lisa, his clock and Lisa's clock both read $t=0$. After a Lab time $2\pi/\omega$ has passed, this Lab observer will encounter Selma, and his clock will lag hers by $\tau_{\text{Selma}}(1-1/\gamma)$ due to time dilation. As above we can argue that τ_{Selma} is greater than Lisa's clock reading by *Disc*, and hence $\tau_{\text{Selma}}=2\pi/(\omega\gamma)+Disc=2\pi\gamma/\omega$. The amount by which the Lab observer's clock lags Selma's is therefore $(2\pi\gamma/\omega)(1-1/\gamma)$. Since Selma's clock leads Lisa's clock by *Disc*, this means that the Lab clock will lead Lisa's clock by

$$Disc-\frac{2\pi\gamma}{\omega}\left(1-\frac{1}{\gamma}\right)=\frac{2\pi}{\omega}\left(1-\frac{1}{\gamma}\right). \tag{8}$$

This lead is precisely what the Lab observer must observe since, due to time dilation, Lisa's clock has been ticking slowly, relative to Lab clocks, by the factor γ, as it has moved through the Lab by an angle of 2π.

3. Synchronization around the ring versus comoving synchronization

We have described Method 4 synchronization as being carried out with light signals propagating on a circular path, yet we have treated the observers on Lisa's ring (aside from the discontinuity) as if they were special relativity observers. More specifically, we have claimed that those observers, from Lisa to Selma, would measure the same time dilation as would observers in a properly synchronized inertial reference frame. Here we give a detailed justification for this. We show that the Method 4 synchronization of two nearby observers on Lisa's ring differs negligibly from the synchronization of a momentarily comoving inertial reference frame.

To do this we consider, just as we did in our discussion of Method 4 synchronization, Lisa and an observer friend at $\Delta\phi$. As in the earlier discussion, event 2 will be the reception of a photon by the friend, and the emission of the return photon. Let us invoke a Cartesian spatial coordinate system $\{x,y\}$ in the Lab, as shown in Fig. 5, with origin at Lisa at the

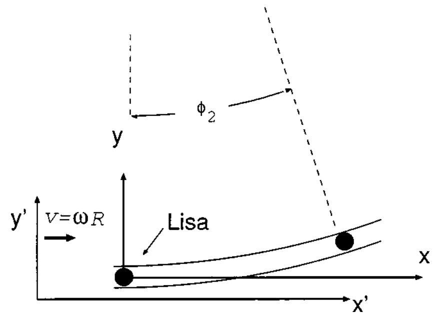

Fig. 5. Method 4 synchronization (as viewed in the Lab frame) and a momentarily comoving inertial reference frame.

moment $(t=0)$ of event 1, and with the x axis in the direction of the emitted photon. The Lab coordinates of event 2 are then

$$t=t_2, \quad x=R\sin\phi_2, \quad y=R(1-\cos\phi_2), \tag{9}$$

where t_2 and ϕ_2 are given in Eq. (2). We now invoke an inertial reference frame $\{t',x',y'\}$ that is momentarily comoving with Lisa. That is, at event 1 this inertial frame is moving in the positive x direction with speed $v=\omega R$, so that instantaneously Lisa is at rest in this frame. By a straightforward Lorentz transformation the time coordinates of event 2 in this momentarily comoving frame are

$$\begin{aligned} t' &= \gamma\left(t+\frac{v}{c^2}x\right)=\gamma\left(t_2+\frac{v}{c^2}R\sin\phi_2\right)\\ &= \frac{R}{c}\gamma\Delta\phi\left(\frac{1}{1+v/c}+\frac{v/c}{\Delta\phi}\sin\left(\frac{\Delta\phi}{1+v/c}\right)\right). \end{aligned} \tag{10}$$

The setting on the clock of Lisa's friend at event 2 is given in Eq. (4). It follows that this setting will lead the setting t' in the inertial frame by $\Delta t \equiv (R/c)\gamma\Delta\phi - t'$, or

$$\Delta t=\frac{v/c}{1+v/c}\frac{R}{c}\gamma\Delta\phi\left(1-\frac{1+v/c}{\Delta\phi}\sin\left(\frac{\Delta\phi}{1+v/c}\right)\right). \tag{11}$$

Suppose Lisa's ring is occupied by N equally spaced observers (with the last observer, Selma, at the same position as Lisa), then $\Delta\phi=2\pi/N$. In the limit that N is very large

$$\begin{aligned} \Delta t &= \frac{v/c}{1+v/c}\frac{R}{c}\gamma\frac{2\pi}{N}\left(1-\frac{N(1+v/c)}{2\pi}\right.\\ &\quad\left.\times\sin\left(\frac{2\pi}{N(1+v/c)}\right)\right) \xrightarrow{N\to\infty} \mathcal{O}(N^{-3}). \end{aligned} \tag{12}$$

If the synchronization of Lisa's ring were carried out from Lisa to Selma, by using momentarily comoving inertial frames, the setting of Selma's clock would be different by only $N\Delta t=\mathcal{O}(N^{-2})$ from the setting arrived at with Method 4. In the limit that N is large (i.e., in the limit that $\Delta\phi$ is small), this is negligible.

[1]A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton U.P., Princeton, 1975), pp. 6 and 7 and 138–140.

[2]T. A. Weber, ''Measurements on a rotating frame in relativity, and the Wilson and Wilson experiment,'' Am. J. Phys. **65** (10), 946–953 (1997).

[3]O. Grøn, ''Relativistic description of a rotating disk,'' Am. J. Phys. **47** (10), 869–876 (1975).

[4]C. Giannoni and O. Grøn, ''Rigidly connected accelerated clocks,'' Am. J. Phys. **47** (5), 431–435 (1979).

[5]It is worth pointing out that many (most?) special relativity paradoxes exploit confusion about simultaneity. Clock synchronization can be considered a specific application of defined simultaneity.

[6]A. Einstein, ''On the electrodynamics of moving bodies,'' in *The Principle of Relativity* (Dover, New York, 1952), pp. 37–55.