Zero time dilation in an accelerating rocket

Ronald P. Gruber
3318 Elm Street, Oakland, California 94609

Richard H. Price
Department of Physics, University of Utah, Salt Lake City, Utah 84112

(Received 31 January 1997; accepted 9 April 1997)

We give a variation of the twin paradox of special relativity in which the relationship of acceleration of the rocket twin and time dilation is clarified.

Numerous articles have struggled with the student misconception that in the classical relativistic twin paradox, the asymmetric aging is caused by acceleration. In most introductory physics texts it is correctly pointed out that the viewpoints of the stay-at-home twin and the rocket twin are not equivalent; the acceleration of the rocket twin distinguishes her from that of the unaccelerated stay-at-home twin. There is a danger that students can infer that the acceleration is in some sense the direct cause of the aging. Certain texts are worse than others in this regard. In a popular calculus-based textbook,[1] for example, the traveler twin has aged less than the stay-at-home twin ''because her bodily processes slowed down during her travels in space.'' Later, it is pointed out that there is an asymmetry in the motion of the twins. It is the rocket twin's ''experience of forces when her spaceship turned around, with [the stay-at-home] twin not subject to such forces.'' To some students this would certainly be an invitation to understand that the force necessary to cause the acceleration is directly responsible for slowing the aging process.

The relationship of acceleration and differential aging is discussed at some length in the recent article by Debs and Redhead,[2] in which a history of the problem is also presented. The solution offered in that article involves doing away with a definitive meaning for simultaneity in an inertial frame. While this is quite interesting in connection with the logical structure of special relativity, it is of little value to the student encountering the twin paradox for the first time.

For students just starting their study of relativity, a more appropriate article is the recent variation on the twin paradox given by Boughn.[3] In Boughn's version, twins experience differential aging, although their history of accelerations is identical. This certainly helps to dispel the idea of any direct connection between acceleration and differential aging, but even here the acceleration is needed to cause the differential aging, and there is the danger that it could be seen as a direct cause of the differential aging. It would seem, therefore, that the best way to make the case that acceleration per se is not the root of asymmetric aging is to give an example of one without the other. Without acceleration[4] the twins cannot meet a second time. This precludes a twin paradox without acceleration. Here, we give the opposite: a simple twin paradox with acceleration, but (in a limit) no asymmetric aging.

To illustrate our point we consider a rocket undergoing periodic motion as illustrated in Fig. 1, motion that—aside from relativistic considerations—would be simple harmonic motion:

$$x=\frac{V_{\max}}{\omega}\sin\,\omega t, \tag{1}$$

where x and t are the coordinates of a fixed frame on earth. It is clear that the maximum speed achieved by the rocket twin, relative to the earth, is $V_{\max}$. The acceleration of the rocket, i.e., the magnitude of the rocket's 4-acceleration, has a maximum magnitude of $V_{\max}\omega$, which occurs at times $\omega t=\pm\pi/2, \pm 3\pi/2,\ldots$. These relativistic results agree perfectly with the Newtonian answers. This is no surprise; the maximum acceleration occurs when the particle has zero velocity relative to the fixed frame on earth, and the relativistic and nonrelativistic results are therefore the same.

The time ticked by clocks on the rocket, that is, the ''proper time'' τ of the rocket, is related to the earth time t, by $d\tau=dt\sqrt{1-(V_{\max}/c)^2\cos^2\omega t}$. A rocket trip starting and ending at the earth will take an integer number, of half cycles of the oscillatory motion, starting, say at $t=0$ and lasting until $\Delta t=n\pi/\omega$. For such a trip the proper time (i.e., the time measured by the rocket's own clocks) will be

$$\Delta\tau=n\int_0^{\pi/\omega}\sqrt{1-(V_{\max}/c)^2\,\cos^2\omega t}\,dt, \tag{2}$$

so that the ratio of elapsed rocket time to elapsed earth time is given by

$$\frac{\Delta\tau}{\Delta t}=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{1-(V_{\max}/c)^2\,\cos^2\theta}\,d\theta. \tag{3}$$

This integral can be evaluated numerically [either by direct numerical integration, or by the fact that the integral is $E(V_{\max}/c)$, where E is the complete Legendre elliptic function of the second kind]. The ratio of $\Delta\tau$ to Δt is shown in

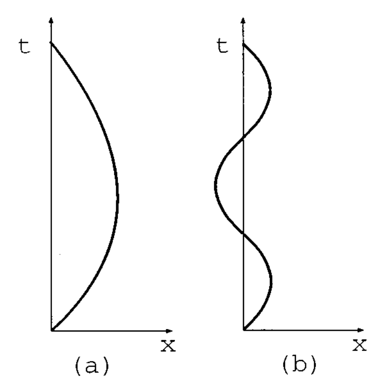

Fig. 1. Two possible rocket-twin worldlines with the same starting and stopping times and the same maximum velocity $V_{\max}$, and hence the same time dilation. The world line in (b) has maximum acceleration three times that for the world line in (a).

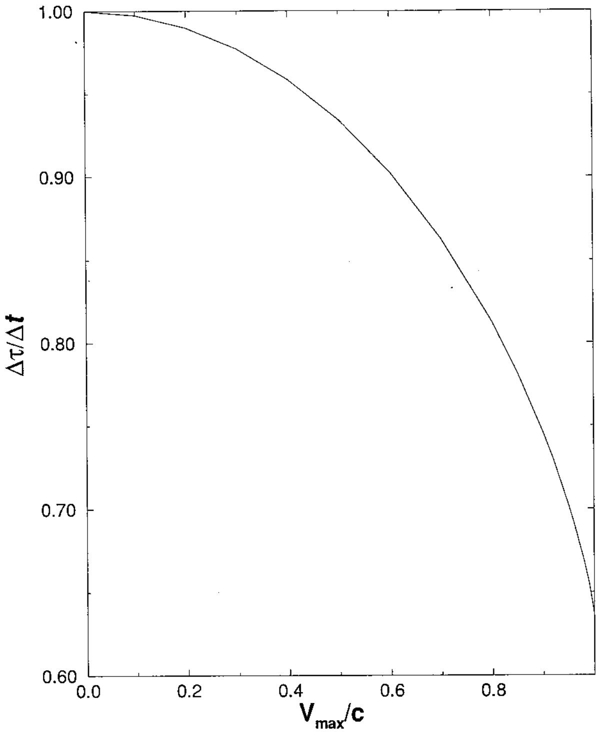

Fig. 2. The time dilation (aging of rocket twin/aging of earth twin) is shown as a function of the maximum velocity V_{max}, for the harmonic motion described in the text.

Fig. 2. It varies from unity (for $V_{max}=0$) to $2/\pi=0.6366...$ (for V_{max} approaching c). What is crucial to note is that this time-dilation effect is independent of ω! The maximum acceleration is $V_{max}\omega$ (and, from dimensional considerations, *any* measure of acceleration will be proportional to $V_{max}\omega$). It follows that time dilation and acceleration can be chosen independently.

The relationship, or lack of one, between the acceleration and the time dilation may be best seen with a numerical example. Let us suppose that for medical reasons the rocket twin chooses a maximum acceleration of g, the familiar free-fall acceleration at the earth surface. If she makes a simple one-dimensional one year trip (six months out, six months back) we calculate:

$$\omega=\pi/1\ \text{yr}=9.94\times10^{-8}\ \text{s}^{-1},\quad V_{max}/c=g/\omega c=0.329. \tag{4}$$

From Eq. (3) we find that $\Delta\tau/\Delta t=0.97$, so that 3% of the rocket twin's year, or about 11 days, will be ''lost'' as seen by the earthbound twin. If, on the other hand, the rocket twin went back and forth (as in Fig. 1) many times, the answer would be very different. Had she made 100 ''legs'' to the trip (that is, 100 half cycles rather than just one) her ω would be larger by a factor of 100, and her V_{max}/c, therefore, smaller by a factor of 100. A computation with Eq. (2) shows that in this case her time dilation would be $\Delta\tau/\Delta t=0.999\,997$. She would rejoin her earthbound sister younger than her twin only by a little more than a minute.

One of us (RHP) gratefully acknowledges support from the National Science Foundation under Grant No. PHY-9507719.

[1]R. A. Serway, *Principles of Physics* (Saunders, Philadelphia, 1994), p. 266.
[2]T. A. Debs and M. L. G. Redhead, ''The 'twin paradox' and the conventionality of simultaneity,'' Am. J. Phys. **64**, 384–392 (1996).
[3]S. P. Boughn, ''The case of the identically accelerated twins,'' Am. J. Phys. **57**, 791–793.
[4]This is only true in the absence of gravitational fields. In the curved space–time of general relativity, unaccelerated worldlines (space–time geodesics) can have multiple crossins.

THE ROLE OF THE EXPERIMENTAL PHYSICIST

My role has been that of an experimental physicist who, by observation and measurement of the properties and operation of the physical world, supplies the data that may lead to the formulation of conceptual structures. The consistency of the consequences of a conceptual structure with the data of physical experiment determines the validity of that structure as a description of the physical universe. Our early predecessors observed Nature as she displayed herself to them. As knowledge of the world increased, however, it was not sufficient to observe only the most apparent aspects of Nature to discover her more subtle properties; rather, it was necessary to interrogate Nature and often to compel Nature, by various devices, to yield an answer as to her functioning. It is precisely the role of the experimental physicist to arrange devices and procedures that will compel Nature to make a quantitative statement of her properties and behavior.

Polykarp Kusch, ''The magnetic moment of the electron'' (Nobel Lecture, December 12, 1955, reprinted in *Nobel Lectures, Physics*, Vol. 3, 1942–1962, Elsevier Amsterdam, 1964).