An alternative derivation of the equations of motion of the relativistic (an)harmonic oscillator

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The equations of motion of the relativistic (an)harmonic oscillator are derived based on an alternative Lagrangian formalism of relativistic mechanics using the proper time as the evolution parameter.

I. INTRODUCTION

The simple harmonic oscillator and its extension to the relativistic case are important subjects in physics. They are usually employed as the basis for modeling more complicated motion. Most of the students in physics are familiar with the simple harmonic oscillator, but not its relativistic extension. That the relativistic extension of simple harmonic motion had not received complete treatment, as compared to the relativistic generalization of constant acceleration, was pointed out by an interesting article in this journal.[1] In that article, the proper-time equations of motion of the relativistic (an)harmonics oscillator were derived based upon the usual Lagrangian formalism of relativistic mechanics. Moreover, the proper-time relativistic motion was analyzed in terms of an effective potential energy in an analogy with classical mechanics.

The present article presents an alternative approach to the relativistic extension of simple harmonic motion, based on an alternative Lagrangian formalism of relativistic mechanics which has been recently developed in terms of the proper time as the evolution parameter.[2] This alternative Lagrangian formalism of relativistic mechanics is closely analogous to the Lagrangian formalism of classical mechanics. Moreover, this alternative Lagrangian formalism provides the conceptual foundations for Schrödinger-like formalism of relativistic quantum mechanics.[3] Since some readers might be unfamiliar with this alternative Lagrangian formalism of relativistic mechanics, we first recapitulate this alternative Lagrangian formalism. Then, this alternative Lagrangian formalism is applied to the relativistic extension of the simple harmonic oscillator. The proper-time equations of relativistic motion are shown as an immediate consequence of this alternative Lagrangian formalism.

II. LAGRANGIAN FORMALISM OF RELATIVISTIC MECHANICS IN TERMS OF THE PROPER-TIME EVOLUTION PARAMETER

We start by considering the general form of the Lagrangian for a relativistic material particle as $L(\mathbf{x},\acute{\mathbf{x}})$ in terms of the proper-time τ as the evolution parameter. It should be noted that the symbol $\acute{\mathbf{x}}$ in $L(\mathbf{x},\acute{\mathbf{x}})$ is the differentiation of position coordinate $\mathbf{x}$ with respect to the proper time τ. The proper time τ for a material particle is related to the coordinate time t by $dt=\gamma d\tau$, where $\gamma\equiv(1-v^2/c^2)^{-1/2}$, v is the speed of the particle, and c is the speed of light. Thus, $\mathbf{u}\equiv\acute{\mathbf{x}}=dx/d\tau=(dx/dt)(dt/d\tau)=\gamma\dot{\mathbf{x}}=\gamma\mathbf{v}$.

Similar to the manifestly Lorentz-covariant Lagrangian formulation of relativistic mechanics,[4–7] Hamilton's principle is expressed as

$$\delta\int_1^2 L(\mathbf{x},\acute{\mathbf{x}})d\tau=0, \tag{1}$$

with the variation in the world line between the two fixed end points, $1=(ct(1),\mathbf{x}(1))$ and $2=(ct(2),\mathbf{x}(2))$. Then, proceeding with the variation, we have

$$\int_1^2\sum_j\left(\frac{\partial L}{\partial x_j}\,\delta x_j+\frac{\partial L}{\partial \acute{x}_j}\,\delta\acute{x}_j\right)d\tau=0. \tag{2}$$

In general, the integration parameter τ depends on the motion of the particle along the world line in the variation. That is, the integration parameter τ in Eq. (2) is different along each world line. The variation in the world line also alters the integration limits of the two end points $\tau(1)$ and $\tau(2)$. It should be emphasized that the variation is taken so as to keep the two end points fixed, i.e., $\delta x_\mu(1)=0$ and $\delta x_\mu(2)=0$ ($\mu=0,1,2,3$), not to keep the values of integration limit $\tau(1)$ and $\tau(2)$ fixed. The variation operation and the proper-time differentiation are interchangeable, that is,

$$\delta\acute{x}_j=\delta\left(\frac{dx_j}{d\tau}\right)=\frac{d}{d\tau}\,\delta x_j,\quad j=1,2,3. \tag{3}$$

Then, Eq. (2) becomes, after integration by parts,

$$\int_1^2\sum_j\left(\frac{\partial L}{\partial x_j}-\frac{d}{d\tau}\frac{\partial L}{\partial\acute{x}_j}\right)\delta x_j d\tau+\sum_j\frac{\partial L}{\partial x_j}\,\delta x_j(2)$$
$$-\sum_j\frac{\partial L}{\partial x_j}\,\delta x_j(1)=0. \tag{4}$$

Since $\delta x_j(1)=0$ and $\delta x_j(2)=0$, Eq. (4) is reduced to

$$\int_1^2\sum_j\left(\frac{\partial L}{\partial x_j}-\frac{d}{d\tau}\frac{\partial L}{\partial\acute{x}_j}\right)\delta x_j d\tau=0. \tag{5}$$

Since the δx_j are independent variations, we have the Lagrange equations of relativistic motion

$$\frac{\partial L}{\partial x_j}-\frac{d}{d\tau}\frac{\partial L}{\partial\acute{x}_j}=0,\quad j=1,2,3. \tag{6}$$

Since the canonical momentum $p_j^c\equiv\partial L/\partial\acute{x}_j$, the Lagrange equations of relativistic motion are also expressed as

$$\acute{p}_j^c=\frac{\partial L}{\partial x_j},\quad j=1,2,3. \tag{7}$$

The relativistic Hamiltonian is defined from the relativistic Lagrangian by

$$H(x_j, p_j^c) = \sum_j p_j^c \acute{x}_j - L(x_j, \acute{x}_j). \tag{8}$$

The total differential of H is

$$dH = \sum_j \left(\frac{\partial H}{\partial x_j} dx_j + \frac{\partial H}{\partial p_j^c} dp_j^c \right). \tag{9}$$

From Eq. (8), we have also

$$dH = \sum_j \left(\acute{x}_j dp_j^c + p_j^c d\acute{x}_j - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial \acute{x}_j} d\acute{x}_j \right). \tag{10}$$

The second and fourth terms in the parentheses in Eq. (10) cancel out, since $p_j^c \equiv \partial L / \partial \acute{x}_j$. Consequently, from Eqs. (6) and (10) we have

$$dH = \sum_j (\acute{x}_j dp_j^c - \acute{p}_j^c dx_j). \tag{11}$$

Comparing Eqs. (9) and (11), the independence of the variations gives the Hamilton equations of relativistic motion:

$$\frac{dx_j}{d\tau} = \frac{\partial H}{\partial p_j^c}, \quad j = 1,2,3, \tag{12a}$$

and

$$\frac{dp_j^c}{d\tau} = -\frac{\partial H}{\partial x_j}, \quad j = 1,2,3. \tag{12b}$$

Using the Poisson bracket

$$\{M,N\} \equiv \sum_j \left[\frac{\partial M}{\partial x_j} \frac{\partial N}{\partial p_j^c} - \frac{\partial M}{\partial p_j^c} \frac{\partial N}{\partial x_j} \right], \tag{13}$$

the Hamilton equations of motion Eqs. (12a) and (12b) are expressed as

$$\frac{dx_j}{d\tau} = \{x_j, H\}, \quad j = 1,2,3, \tag{14a}$$

and

$$\frac{dp_j^c}{d\tau} = \{p_j^c, H\}, \quad j = 1,2,3. \tag{14b}$$

Now, consider a relativistic particle moving in a conservative force field of potential $V(\mathbf{x})$. According to special relativity, the energy–momentum relation is given as

$$(E - V)^2 - \mathbf{p}^2 c^2 = m^2 c^4, \tag{15}$$

where $\mathbf{p} = m\mathbf{u}$, with $\mathbf{u} \equiv d\mathbf{x}/d\tau$.

If we define a quantity K as

$$K \equiv \frac{E^2 - m^2 c^4}{2mc^2}, \tag{16}$$

then Eq. (15) can be rewritten as

$$K = \frac{\mathbf{p}^2}{2m} + V_{\text{eff}}, \tag{17}$$

where

$$V_{\text{eff}} \equiv \frac{2EV - V^2}{2mc^2}. \tag{18}$$

The quantity K can be thought of as an energy inclusive of the Newton-like relativistic kinetic energy $\frac{1}{2}m\mathbf{u}^2$ and the *effective* potential energy V_{eff}.

We choose the relativistic Lagrangian as

$$L = \tfrac{1}{2} m\mathbf{u}^2 - V_{\text{eff}}. \tag{19}$$

It should be emphasized that the total energy E in the definition of the effective potential energy must not be considered, in advance, as a *known* function depending upon the variables $\mathbf{x}$ and $\mathbf{u}$ *explicitly*, though the total energy indeed contains the rest-mass energy, the kinetic energy, and the potential energy. For a conservative system, the total energy E is just a given constant. With a given total energy, the relationship between the kinetic energy and the potential energy of the particle is determined from the Lagrange equations of motion with a suitable Lagrangian. The effective potential V_{eff} does not depend on $\mathbf{u}$ *explicitly*, because the potential V is independent of $\mathbf{u}$. From the definition of canonical momentum, we have $p_j^c = \partial L / \partial \acute{x}_j = \partial L / \partial u_j = p_j$. From the Lagrange equations of relativistic motion, Eq. (6), we have

$$\frac{dp_j}{d\tau} = -\frac{(E - V)}{mc^2} \frac{\partial V}{\partial x_j}, \quad j = 1,2,3. \tag{20}$$

Since $dt = \gamma d\tau$, Eq. (20) becomes

$$\frac{dp_j}{dt} = -\frac{\partial V}{\partial x_j}, \quad j = 1,2,3, \tag{21}$$

provided that $E - V = \gamma mc^2$. Equation (21) is just the equation of motion in special relativity, $(d/dt)\mathbf{p} = -\nabla V$. Also, the relation $E - V = \gamma mc^2$ is consistent with the given energy–momentum relation Eq. (15). That is, the energy–momentum relation is just a consequence of the Lagrange equations of motion with the chosen Lagrangian. From the chosen relativistic Lagrangian and the definition of the relativistic Hamiltonian Eq. (8), we have

$$H = \frac{\mathbf{p}^2}{2m} + V_{\text{eff}}. \tag{22}$$

The relativistic Hamiltonian H is not the total energy E, but it is equal to $K = (E^2 - m^2c^4)/2mc^2$.

Furthermore, consider generally a particle of charge e moving in external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. According to special relativity, the relativistic energy–momentum relation is

$$(E - e\Phi)^2 - \left(\mathbf{P} - \frac{e}{c} \mathbf{A} \right)^2 c^2 = m^2 c^4. \tag{23}$$

Here, the scalar potential $\Phi(\mathbf{x},t)$ and the vector potential $\mathbf{A}(\mathbf{x},t)$ form a Lorentz-covariant four-vector. The four-vector $(\Phi,\mathbf{A})$ is related to electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ by

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \tag{24}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{25}$$

It should be noted that E is the *total* energy of the charged particle, whereas, $\mathbf{E}$ is the electric field. By using the definition of the quantity K, Eq. (23) can be rewritten as

$$K=\frac{\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)^2}{2m}+V_{\mathrm{eff}},\tag{26}$$

where $V_{\mathrm{eff}}\equiv(2EV-V^2)/2mc^2$, and $V\equiv e\Phi$.

Now, we choose the relativistic Lagrangian as

$$L=\frac{1}{2}m\mathbf{u}^2+\frac{e}{c}\mathbf{u}\cdot\mathbf{A}-V_{\mathrm{eff}}.\tag{27}$$

Then, the canonical momentum $\mathbf{p}^c$ conjugate to the position coordinate $\mathbf{x}$ is

$$p_j^c\equiv\frac{\partial L}{\partial u_j}=p_j+\frac{e}{c}A_j=P_j,\quad j=1,2,3.\tag{28}$$

Then, from the given relativistic Lagrangian and the definition of the relativistic Hamiltonian, we have

$$H=\frac{\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)^2}{2m}+V_{\mathrm{eff}}.\tag{29}$$

Therefore, the relativistic Hamiltonian H is equal to $K=(E^2-m^2c^4)/2mc^2$.

Moreover, from the Lagrange equations of relativistic motion, we have

$$\frac{d}{d\tau}\left(p_j+\frac{e}{c}A_j\right)=-\frac{(E-V)}{mc^2}\frac{\partial V}{\partial x_j}+\left(\frac{e}{c}\right)\frac{\partial}{\partial x_j}(\mathbf{u}\cdot\mathbf{A}),\qquad j=1,2,3.\tag{30}$$

The total differential $(d/d\tau)A_j$ consists of two parts: the change of the vector potential with time at a fixed point in space, and the change due to motion of the particle from one point in space to another, that is,

$$\frac{dA_j}{d\tau}=\frac{dA_j}{dt}\frac{dt}{d\tau}=\gamma\left[\frac{\partial A_j}{\partial t}+(\mathbf{v}\cdot\nabla)A_j\right],\quad j=1,2,3.\tag{31}$$

Since $dt=\gamma d\tau$, and $\mathbf{u}=\gamma\mathbf{v}$, Eqs. (30) and (31) give

$$\frac{dp_j}{dt}=e\left(-\frac{\partial\Phi}{\partial x_j}-\frac{1}{c}\frac{\partial A_j}{\partial t}\right)+\frac{e}{c}\left(\mathbf{v}\cdot\frac{\partial\mathbf{A}}{\partial x_j}-\mathbf{v}\cdot\nabla A_j\right),\qquad j=1,2,3,\tag{32}$$

provided that $E-V=\gamma mc^2$. By using Eqs. (24) and (25), Eq. (32) is reduced to

$$\frac{d\mathbf{p}}{dt}=e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right).\tag{33}$$

This equation is just the equation of motion of a relativistic particle in electromagnetic fields in accordance with the Lorentz force law in special relativity. The alternative relativistic Lagrangian formalism is consistent with special relativity. The nonmanifestly covariant electromagnetic force law Eq. (33), which holds in all inertial frames, is derived from the relativistic Lagrangian Eq. (27), the mathematical form of which is not manifestly Lorentz covariant. Moreover, the Lagrange equations of relativistic motion, Eq. (6), and the Hamilton equations of relativistic motion, Eq. (14), are not manifestly Lorentz covariant. This alternative relativistic Lagrangian formalism provides another example showing that the mathematical forms of laws of physics, though not manifestly Lorentz covariant, may indeed be invariant with respect to all inertial frames.

In the low-speed limit, that is, for classical mechanics, the relativistic Lagrangian Eq. (27) reduces to the classical Lagrangian

$$L=\frac{1}{2}m\mathbf{v}^2+\frac{e}{c}\mathbf{v}\cdot\mathbf{A}-e\Phi,\tag{34}$$

because $\mathbf{p}$ is reduced to $m\mathbf{v}$, and V_{eff} is reduced to $e\Phi$, due to $|e\Phi|\ll E\sim mc^2$. Also, the relativistic Hamiltonian Eq. (29) is reduced to the classical Hamiltonian

$$H=\frac{\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)^2}{2m}+e\Phi,\tag{35}$$

with $\mathbf{P}=m\mathbf{v}+(e/c)\mathbf{A}$. Consequently, the equations of motion for a low-speed particle in the electromagnetic fields,

$$\frac{d}{dt}(m\mathbf{v})=e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right),\tag{36}$$

are obtained from the classical Lagrangian Eq. (34) and the Lagrange equations of motion in classical mechanics,[4,5]

$$\frac{\partial L}{\partial x_j}-\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_j}=0,\quad j=1,2,3.\tag{37}$$

In the low-speed limit, since $\gamma=1$, we have $d\tau=dt$, and thus $\acute{x}_j=dx_j/d\tau=dx_j/dt=\dot{x}_j$. Therefore, the equations of nonrelativistic motion, Eq. (37), are the low-speed limit of the equations of relativistic motion, Eq. (6). This alternative Lagrangian formalism for relativistic motion contains the Lagrangian formalism for nonrelativistic motion as the low-speed limit. This alternative Lagrangian formalism of relativistic mechanics is analogous to the Lagrangian formalism of classical mechanics, comparing Eqs. (6), (27), and (29) with Eqs. (37), (34), and (35), respectively.

III. THE EQUATIONS OF MOTION OF THE RELATIVISTIC (AN)HARMONIC OSCILLATOR

This alternative Lagrangian formalism is applied to the simple harmonic oscillator to obtain the equations of relativistic motion. We consider the relativistic motion of a particle of rest mass m in a one-dimensional harmonic oscillator potential $V(x)=\frac{1}{2}kx^2$, where k is a constant. From the definition of the effective potential energy Eq. (18), we have

$$V_{\mathrm{eff}}=\frac{Ekx^2-(\frac{1}{2}kx^2)^2}{2mc^2}.\tag{38}$$

Also, from the chosen Lagrangian Eq. (19), we have the Lagrangian

$$L=\frac{1}{2}m\acute{x}^2-\frac{Ekx^2-(\frac{1}{2}kx^2)^2}{2mc^2}.\tag{39}$$

The canonical momentum is

$$p=\frac{\partial L}{\partial\acute{x}}=m\acute{x}=mu=\gamma mv.\tag{40}$$

Then, according to the Lagrange equations, Eq. (6), we have

$$\frac{dp}{d\tau}+\frac{(E-\frac{1}{2}kx^2)}{mc^2}kx=0. \tag{41}$$

Since $dt=\gamma d\tau$, this equation becomes the equation of relativistic motion of a particle in the potential $V(x)=\frac{1}{2}kx^2$,

$$\frac{dp}{dt}+kx=0, \tag{42}$$

provided that

$$E-\tfrac{1}{2}kx^2=\gamma mc^2. \tag{43}$$

This equation means that the total energy E is the sum of the rest mass energy mc^2, the relativistic kinetic energy $T=(\gamma-1)mc^2$, and the potential energy $\frac{1}{2}kx^2$; that is

$$E=mc^2+T+\tfrac{1}{2}kx^2. \tag{44}$$

According to the Hamiltonian definition in Eq. (7), we have $H=p^2/2m+V_{\rm eff}$. The Hamiltonian (*effective total energy*) H is the sum of the *effective* kinetic energy $p^2/2m=\frac{1}{2}mu^2$ and the *effective* potential energy. The Hamiltonian is not equal to the total energy E, but is equal to $(E^2-m^2c^4)/2mc^2$ which is a constant of the motion. By setting $k\equiv m\omega^2$ and $\gamma_0\equiv E/mc^2$, Eq. (43) is rewritten as

$$\frac{dt}{d\tau}+\frac{\omega^2x^2}{2c^2}=\gamma_0, \tag{45}$$

and Eq. (41) is rewritten as

$$\frac{d^2x}{d\tau^2}+\omega^2x\,\frac{dt}{d\tau}=0. \tag{46}$$

These equations of motion, Eqs. (46) and (45), are the proper-time equations of motion of the relativistic (an)harmonic oscillator, Eqs. (15) and (16), as given in Ref. 1 and solved therein. The parametrized Lagrangian formalism provides a simple and direct method to obtain the proper-time equation of motion. Finally, it should be emphasized that the concepts of effective energies are applicable not only to the special case of the relativistic (an)harmonic oscillator, but also to the relativistic extension of classical mechanics generally.

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MINUTE PARTICULARS

When William Blake once gave his thinking attention to the perennial cry of those who justified their deeds in the name of the common good, he had to conclude:

> He who would do good to another must do it in Minute Particulars. General Good is the plea of the scoundrel, hypocrite, flatterer; for Art and Science cannot exist but in minutely organized Particulars.

The deepest, most pervasive theme of American educationism is the rejection of minutely organized particulars for the sake of vaguely appreciated generalities. If the former are the substance of Art and Science, of what are the latter the substance?

Richard Mitchell, *The Graves of Academe* (Little, Brown and Company, Boston, 1981), p. 206.