Twin paradox: A complete treatment from the point of view of each twin

Robert Perrin
Phillips Academy, Andover, Massachusetts 01810
(Received 22 March 1978; accepted 4 September 1978)

A specific round trip situation is set up, and is worked through from the point of view of each twin. The gravitational field equations, and geodesic equations of motion, are solved in the traveling twin's reference frame, in order to determine the time elapsed on the Earth during the periods of acceleration. The equality of the results obtained by each twin is explicitly exhibited.

I. INTRODUCTION

When the twin paradox is discussed in courses or textbooks on special relativity, the treatment usually ends with statements akin to the following: "Although there is no need to invoke general relativity theory in explaining the twin paradox, the student may wonder what the outcome of the analysis would be if we knew how to deal with accelerated reference frames. We could then use (the) space ship as our reference frame. . . . We would find that we must have a gravitational field in this frame to account for the accelerations. . . . If, as required in general relativity, we then compute the frequency shifts of light in this gravitational field, we come to the same conclusion as in special relativity."[1] If the student were to pursue the situation further, and actually attempt to seek out a reference in which the general theory is used to work the problem through, he would find that such treatments are not readily available. We have only been able to find one textbook on general relativity that does have such a treatment, and that is *The Theory of Relativity* by Moller.[2,3]

It is the purpose of this article to modify and expand on Moller's treatment. It is felt that there are three aspects of his solution that warrant further consideration. First, Moller never makes use of the gravitational field equations. He solves the problem by transforming the Lorentz metric into a form that is valid in arbitrary accelerated reference frames. Our solution will take the field equations as its starting point. Second, although Moller's results are valid for arbitrary values of the acceleration, he only shows consistency with the special theory in the limit of infinite accelerations. We shall show consistency for the general case. Finally, although Moller's metric has the virtue that its spatial part is Euclidean, it leads to rather complicated geodesic equations of motion. The form of the metric used in our solution leads to equations of motion that are easily soluble.

II. RESULTS FROM THE SPECIAL THEORY

We denote the reference frame of the earth twin by S, and that of the traveling twin by $\bar{S}$. All unbarred quantities refer to S, and barred quantities to $\bar{S}$. The round trip we consider is the following: The ship leaves the earth with an acceleration $\bar{a} = g$, which is constant in $\bar{S}$. It maintains this acceleration until it reaches a velocity v relative to the earth. We denote its distance from the earth at this instant by d. The acceleration is then cut off, and the ship travels a further distance L with constant velocity v. An acceleration $\bar{a} = -g$ is then instituted, and is maintained until the ship has reversed its direction of motion, and is traveling towards the earth with velocity $-v$. It then retraces the distance L with constant speed v, the final distance d with acceleration $\bar{a} = g$, and ends up at rest on the earth.

From the point of view of the earth twin in S, the outward trip appears as schematically shown in Fig. 1. Since S is an inertial frame, the results of special relativity apply. During phase (1), the acceleration in S is

$$a = \frac{dv}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} g, \tag{1}$$

whence,

$$dt = \frac{dv}{g(1 - v^2/c^2)^{3/2}}, \tag{2}$$

$$t_1 = \frac{1}{g}\int_0^v \left(1 - \frac{v^2}{c^2}\right)^{-3/2} dv = \frac{v/g}{(1 - v^2/c^2)^{1/2}}. \tag{3}$$

The time during phase (2) is $t_2 = L/v$, and during phase (3) it is the same as during phase (1). Therefore, the total elapsed time in S is

$$T = \frac{4v/g}{(1 - v^2/c^2)^{1/2}} + \frac{2L}{v}. \tag{4}$$

To compute the time elapsed in $\bar{S}$, the earth twin uses the standard time dilation result. During phase (1), this gives

$$d\bar{t} = (1 - v^2/c^2)^{1/2} dt = \frac{dv}{g(1 - v^2/c^2)}, \tag{5}$$

$$\bar{t}_1 = \frac{1}{g}\int_0^v \left(1 - \frac{v^2}{c^2}\right)^{-1} dv = \frac{c}{2g}\ln\left(\frac{1 + v/c}{1 - v/c}\right), \tag{6}$$

with an identical result holding during phase (3). During phase (2), the result is

$$\bar{t}_2 = t_2\,(1 - v^2/c^2)^{1/2} = (L/v)\,(1 - v^2/c^2)^{1/2}. \tag{7}$$

Therefore, the total elapsed time in $\bar{S}$ is

$$\bar{T} = \frac{2c}{g}\ln\left(\frac{1 + v/c}{1 - v/c}\right) + \frac{2L}{v}\,(1 - v^2/c^2)^{1/2}. \tag{8}$$

We shall later need the distance d expressed in terms of v and g. Using (2), we have

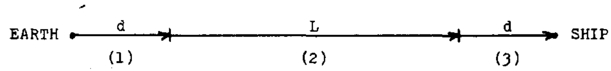

Fig. 1. Schematic representation of the outward trip from the point of view of the earth twin in S.

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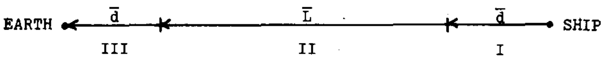

Fig. 2. Schematic representation of the outward trip from the point of view of the traveling twin in $\bar S$.

$$d = \int_0^v v\,dt = \frac{1}{g}\int_0^v v\left(1-\frac{v^2}{c^2}\right)^{-3/2} dv = \frac{c^2}{g}\left(\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right). \quad (9)$$

From the point of view of the traveling twin in $\bar S$, the situation appears as shown in Fig. 2. For a time

$$\bar t_{\rm I} = \frac{c}{2g}\ln\left(\frac{1+v/c}{1-v/c}\right), \quad (10)$$

a gravitational field is turned on in the $\bar x$ direction. The classical gravitational potential corresponding to this field is $\phi(\bar x) = g\bar x$. The earth falls freely under the influence of this gravitational field. At the instant the field is turned off, the earth is at a distance

$$\bar d = d(1-v^2/c^2)^{1/2} = (c^2/g)[1-(1-v^2/c^2)^{1/2}], \quad (11)$$

from the ship, and is moving with velocity $-v$. The earth continues to move a further distance

$$\bar L = L(1-v^2/c^2)^{1/2}, \quad (12)$$

at this constant speed v. The elapsed time in $\bar S$ during this second phase is

$$\bar t_{\rm II} = \bar L/v = (L/v)\,(1-v^2/c^2)^{1/2}. \quad (13)$$

During phase III, a gravitational field is turned on in the opposite sense from phase I, and under its influence the earth comes to rest, and subsequently reverses its direction of motion. The duration of this phase is $\bar t_{\rm III} = \bar t_{\rm I}$, where $\bar t_{\rm I}$ is given by (10). The return trip of the earth is identical in all respects to the outward trip. The total elapsed time in $\bar S$ is as given in (8).

To compute the time elapsed in S, the travelling twin can only use the results of special relativity during phase II. During phases I and III, he must make use of the general theory.

The time dilation result gives for the duration of phase II in S,

$$t_{\rm II} = \bar t_{\rm II}\,(1-v^2/c^2)^{1/2} = (L/v)\,(1-v^2/c^2). \quad (14)$$

The total elapsed time in S is

$$T = 2t_{\rm I} + 2t_{\rm III} + (2L/v)(1-v^2/c^2). \quad (15)$$

A comparison with (4) shows that the general theory must predict

$$t_{\rm I} + t_{\rm III} = \frac{2v/g}{(1-v^2/c^2)^{1/2}} + \frac{Lv}{c^2}. \quad (16)$$

In Sec. III we shall obtain this result.

III. RESULTS FROM THE GENERAL THEORY

We use the notation and sign conventions of Adler, Bazin, and Schiffer.[4] The gravitational field in $\bar S$ occurs as a result of an acceleration of the ship in the $\bar x$ direction. Accordingly, we seek a metric in $\bar S$ of the form

$$ds^2 = e^{\alpha(\bar x)}c^2 d\bar t^2 - e^{\beta(\bar x)} d\bar x^2 - d\bar y^2 - d\bar z^2, \quad (17)$$

where $\alpha(\bar x)$ and $\beta(\bar x)$ are unknown functions of $\bar x$. The Christoffel symbols corresponding to the metric in (17) are

$$\left\{ {0 \atop \mu\nu} \right\} = \frac{1}{2}(\delta_{\mu 0}\delta_{\nu 1} + \delta_{\nu 0}\delta_{\mu 1})\alpha'(\bar x),$$
$$\left\{ {1 \atop \mu\nu} \right\} = \frac{1}{2}\delta_{\mu 0}\delta_{\nu 0}e^{\alpha(\bar x)-\beta(\bar x)}\alpha'(\bar x) + \frac{1}{2}\delta_{\mu 1}\delta_{\nu 1}\beta'(\bar x), \quad (18)$$
$$\left\{ {2 \atop \mu\nu} \right\} = \left\{ {3 \atop \mu\nu} \right\} = 0,$$

where $\delta_{\mu\nu} = 1$ if $\mu = \nu$, and $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$.[5] The free-space field equations $R_{\mu\nu} = 0$ yield the single differential equation

$$\alpha''(\bar x) + (1/2)[\alpha'(\bar x)]^2 - (1/2)\alpha'(\bar x)\beta'(\bar x) = 0 \quad (19)$$

Taking $\alpha'(\bar x) \neq 0$,[6] we can solve this equation for $\beta(\bar x)$. The result is

$$\beta(\bar x) = 2\ln[\alpha'(\bar x)] + \alpha(\bar x) + \ln(A^2), \quad (20)$$

where A is a constant. The metric now has the form

$$ds^2 = e^{\alpha(x)}c^2 d\bar t^2 - A^2[\alpha'(\bar x)]^2 e^{\alpha(\bar x)} d\bar x^2 - d\bar y^2 - d\bar z^2. \quad (21)$$

The function $\alpha(\bar x)$ can be chosen arbitrarily, subject only to the condition $\alpha'(\bar x) \neq 0$. Choosing a different form for the function corresponds to redefining the coordinate $\bar x$. Using

$$\alpha(\bar x) = \bar x/A, \quad (22)$$

gives

$$ds^2 = e^{\bar x/A}(c^2 d\bar t^2 - d\bar x^2) - d\bar y^2 - d\bar z^2. \quad (23)$$

Making use of the classical limit to determine the constant A,

$$g_{00}(\bar x) = e^{\bar x/A} \approx 1 + \bar x/A \approx 1 + 2\phi(\bar x)/c^2 = 1 \pm 2g\bar x/c^2, \quad (24)$$

we obtain as our final form for the metric

$$ds^2 = e^{\pm 2g\bar x/c^2}(c^2 d\bar t^2 - d\bar x^2) - d\bar y^2 - d\bar z^2, \quad (25)$$

with the plus sign holding during phase I, and the minus sign during phase III. The corresponding Christoffel symbols are

$$\left\{ {0 \atop \mu\nu} \right\} = \pm\frac{g}{c^2}(\delta_{\mu 0}\delta_{\nu 1} + \delta_{\nu 0}\delta_{\mu 1}),$$
$$\left\{ {1 \atop \mu\nu} \right\} = \pm\frac{g}{c^2}(\delta_{\mu 0}\delta_{\nu 0} + \delta_{\mu 1}\delta_{\nu 1}), \quad (26)$$
$$\left\{ {2 \atop \mu\nu} \right\} = \left\{ {3 \atop \mu\nu} \right\} = 0.$$

During phases I and III, the earth moves along a geodesic in $\bar S$. The elapsed time in S corresponds to the elapsed proper time along the geodesic. In order to compute this, we must solve the equations of motion

$$\frac{d^2\bar x^\mu}{ds^2} + \left\{ {\mu \atop \alpha\beta} \right\}\frac{d\bar x^\alpha}{ds}\frac{d\bar x^\beta}{ds} = 0. \quad (27)$$

Using (26), these become

$$\frac{d^2\bar t}{ds^2} \pm \frac{2g}{c^2}\left(\frac{d\bar t}{ds}\right)\left(\frac{d\bar x}{ds}\right) = 0,$$

$$\frac{d^2\bar{x}}{ds^2} \pm g\left(\frac{d\bar{t}}{ds}\right)^2 \pm \frac{g}{c^2}\left(\frac{d\bar{x}}{ds}\right)^2 = 0, \tag{28}$$
$$\frac{d^2\bar{y}}{ds^2} = \frac{d^2\bar{z}}{ds^2} = 0.$$

The last two equations have the solutions $\bar{y} = \bar{z} = 0$, since the earth moves in the $\bar{x}$ direction. Adding the second equation to $\pm\, c$ times the first equation gives

$$\frac{d^2(\bar{x} \pm c\bar{t})}{ds^2} \pm \frac{g}{c^2}\left(\frac{d(\bar{x} \pm c\bar{t})}{ds}\right)^2 = 0. \tag{29}$$

These are two uncoupled equations for $\bar{x} \pm c\bar{t}$, and are easily solved to give

$$\bar{x} = \pm\, c^2/2g \ln|(A + gs/c^2)(B \pm gs/c^2)| + C,$$
$$\bar{t} = \pm\, c/2g \ln|(A \pm gs/c^2)/(B \pm gs/c^2)| + D. \tag{30}$$

The constants A, B, C, and D are to be determined from the boundary conditions.

We first require that the parameter s be the arc length along the geodesic. From (25), we have

$$1 = e^{\pm 2g\bar{x}/c^2}\left[c^2\left(\frac{d\bar{t}}{ds}\right)^2 - \left(\frac{d\bar{x}}{ds}\right)^2\right]. \tag{31}$$

Substitution of (30) into (31) gives

$$C = 0,$$
$$|(A \pm gs/c^2)(B \pm gs/c^2)| = -\,(A \pm gs/c^2)(B \pm gs/c^2) \tag{32}$$

For phase I, the boundary conditions at $s = 0$ are

$$\bar{t}(0) = 0, \quad \bar{x}(0) = 0, \quad \frac{d\bar{x}}{d\bar{t}}(0) = 0. \tag{33}$$

These imply $A = 1$, $B = -1$, and $D = 0$. From (30) and (32) there now follow:

$$\bar{x} = c^2/2g \ln[(1 + gs/c^2)(1 - gs/c^2)],$$
$$\bar{t} = c/2g \ln\left(\frac{1 + gs/c^2}{1 - gs/c^2}\right). \tag{34}$$

Inverting the second of these equations gives

$$s = c^2/g \tanh(g\bar{t}/c). \tag{35}$$

This corresponds to Moller's result (8.175).[2] The elapsed proper time along the geodesic is s/c. Evaluating (35) at $\bar{t}_{\rm I}$, as given in (10), we obtain for the elapsed time in S during phase I,

$$t_{\rm I} = s/c = v/g. \tag{36}$$

For phase III we shall use the return portion of the trip, rather than the outbound portion. This will result in algebraically simpler results. We take for two of the boundary conditions at $s = 0$,

$$\bar{t}(0) = 0, \quad \frac{d\bar{x}}{d\bar{t}}(0) = 0. \tag{37}$$

To evaluate the third boundary condition, the value of $\bar{x}(0)$, we use the fact that the spatial separation between the earth and the ship at $\bar{t}(0) = 0$ is $L + 2d$, since both are in the same instantaneous Lorentz frame. The metric with $\bar{t}$ constant reduces to

$$ds^2 = -\,d\bar{l}^2 = -\,e^{-2g\bar{x}/c^2}d\bar{x}^2, \tag{38}$$

so that

$$L + 2d = \int_{\bar{x}(0)}^{0} d\bar{l} = \int_{\bar{x}(0)}^{0} e^{-g\bar{x}/c^2}d\bar{x}$$
$$= c^2/g(e^{-g\bar{x}(0)/c^2} - 1). \tag{39}$$

This gives

$$\bar{x}(0) = -\,c^2/g \ln[g(L + 2d)/c^2 + 1]. \tag{40}$$

The conditions (37) and (40) imply

$$A = g(L + 2d)/c^2 + 1, \quad B = -A, \quad D = 0. \tag{41}$$

From (30) and (32), there now follow:

$$\bar{x} = -\,c^2/2g \ln[(A - gs/c^2)(A + gs/c^2)],$$
$$\bar{t} = -\,c/2g \ln[(A - gs/c^2)/(A + gs/c^2)]. \tag{42}$$

Inverting the second of these gives

$$s = (c^2/g)A \tanh(g\bar{t}/c)$$
$$= (L + 2d + c^2/g)\tanh(g\bar{t}/c). \tag{43}$$

Evaluating (43) at $\bar{t}_{\rm III} = \bar{t}_{\rm I}$, as given in (10), and using (9), we have for the elapsed time in S during phase III

$$t_{\rm III} = \frac{s}{c} = \frac{v}{c^2(L + 2d + c^2/g)}$$
$$= \frac{Lv}{c^2} + \frac{2v/g}{(1 - v^2/c^2)^{1/2}} - v/g \tag{44}$$

Adding (36) and (44) gives

$$t_{\rm I} + t_{\rm III} = \frac{2v/g}{(1 - v^2/c^2)^{1/2}} + \frac{Lv}{c^2}, \tag{45}$$

which is identical to (10). This explicitly exhibits the consistency of the two approaches.

[1]R. Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), p. 207.

[2]C. Moller, *The Theory of Relativity* (Oxford University, New York, 1952).

[3]The crux of the resolution of the "paradox" involves calculating the time that passes on the earth when the travelling twin changes inertial frames, i.e., during the periods of acceleration. Some papers that focus on this aspect of the problem, strictly within the context of special relativity are the following: R. H. Romer, Am. J. Phys. **27**, 131 (1959); G. Builder, *ibid.* **27**, 656 (1959); E. Lowry, *ibid.* **31**, 59 (1963); L. Levi, *ibid.* **35**, 968 (1967); R. A. Muller, *ibid.* **40**, 966 (1972).

[4]R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).

[5]$\delta_{\mu\nu}$, thus defined, is not a tensor.

[6]Otherwise we would have no gravitational field in the classical limit.