The lock and key paradox and the limits of rigidity in special relativity

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According to special relativity, a (longitudinally) moving stick is Lorentz contracted. When it is brought to rest, it must expand to its proper length. But, exactly how this expansion unfolds depends on how the stick is stopped. If the front end hits a brick wall, the rear end must (briefly) continue moving, and the stick contracts even further before expanding; if instead the rear end is suddenly stopped, the front end (briefly) continues, and (surprisingly) the stick overexpands before settling into its proper length. These effects of overexpanding and overcontracting are independent of any classical or molecular elasticity, but are derived entirely from the limits of information travel time imposed by special relativity. I explore these phenomena, inspired by the little known "lock and key" paradox.
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I. THE LOCK AND KEY PARADOX

Consider the following paradox: A U-shaped lock for a safe is fitted by a T-shaped key in their mutual rest frame, but the key is too short to reach all the way into the lock (see Fig. 1). Let's put a button at the deepest part of the lock (point l) and stipulate that the safe opens only when the end of the key k touches l. When the lock and key are at rest, the tip of the key does not reach the button and the money is secure.

Suppose we slam the two together at a relativistic speed. According to the key, the lock has become Lorentz contracted, so the key now easily reaches the button before its back hits the prongs of the lock. According to the lock, however, the key has become Lorentz contracted, and hence is even further from reaching the button. In one frame of reference the button is pressed and the safe opens, whereas in the other frame it doesn't.

Note that this paradox is not an issue of simultaneity, like the barn and ladder paradox.[1] Here, there is an objective event that all observers must agree on—either the button is pressed or it isn't.

The resolution lies in the finite speed with which information can travel through the key. This speed is the speed of sound in the material of the lock and key, but for the sake of argument let's say that sound travels at the speed of light c. The moment the back of the key hits the prongs of the lock, the message is sent along the key at the speed of light to inform the tip that the back has come to a halt and the rest needs to follow suit. In the spirit of examining the limits imposed by special relativity, let's assume that the key is unbreakable and that the moment the tip of the key receives the information that the back has stopped, it will adjust as fast as it can to the proper length—in formal terms, this means that the key has infinite tension. In the meantime, after the back of the key has hit the prongs of the lock, yet before the news has made it to the tip of the key, the tip will continue moving toward the button.

Is it possible that the tip of the key overshoots and hits the button before the information reaches the tip to let it know that it must return to its proper length? If so, the key actually hits the button in both frames of reference and the paradox is resolved. But, if the key extends until it assumes its proper length, and having reached it, stops, then the paradox remains. We will show in Sec. II that the key must overextend (before the information has reached the tip) for all nonzero velocities.

II. OVEREXTENSION AND OVERCOMPRESSION

Consider the following situation. A stick of proper length L flies by you at a tremendous velocity v. As it passes, you reach up and grab the back end of the stick, stopping it instantly [see Fig. 2(a)]. In time the information will reach the front of the stick, and it will eventually assume its proper length in your rest frame. If we assume that the information propagates at the speed of light, the time it takes for the information to travel from the back of the stick to its front is given by $ct=(L/\gamma)+vt$, or

$$t=\frac{L}{(c-v)\gamma}, \tag{1}$$

where $\gamma=1/\sqrt{1-v^2/c^2}$. In that time the front of the stick has traveled a distance

$$vt=\frac{vL}{(c-v)\gamma}. \tag{2}$$

The stick is initially Lorentz contracted and must eventually expand to its proper length. We define the coefficient of overextension to be proportional to the difference between its maximum length and the length it will eventually assume,

$$\eta_E \equiv \frac{1}{L}\left(vt+\frac{L}{\gamma_0}-\frac{L}{\gamma_f}\right), \tag{3}$$

where γ_0 is the initial value of γ, γ_f is the final value of γ, and v is the magnitude of the change in velocity of the stick. In this case,

$$\eta_E=\frac{1}{L}(ct-L)=\sqrt{1-\frac{v^2}{c^2}}\frac{c}{c-v}-1=D-1, \tag{4}$$

where D is the relativistic Doppler shift factor, $D=\sqrt{(1+v/c)/(1-v/c)}$. The coefficient $\eta_E(v)$ is plotted in Fig. 3. Note that the stick always overextends ($\eta_E>0$), and wildly so as it approaches the speed of light. There is no limit on how much the stick can be overextended.

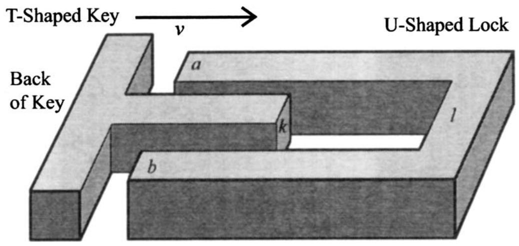

Fig. 1. The lock and key paradox. A T-shaped key fits into a U-shaped lock, but the key is too short to reach all the way into the lock. Points a and b represent the prongs of the lock, k represents the tip of the key, and l represents the deepest part of the lock (where the button is placed).

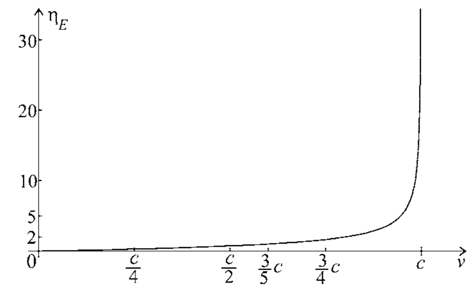

Fig. 3. Plot of the overextension coefficient in Eq. (4). For $v=(3/5)c$, $\eta_E=1$, which corresponds to a maximum length of twice the proper length. η_E approaches infinity as v approaches c, so there is no limit on the theoretical maximum length achievable from this perspective.

Suppose instead that the stick is initially at rest, and we grab the front end and give it such a ferocious pull that it immediately accelerates to a velocity v [see Fig. 2(b)]. It will take time for the news to reach the back of the stick that the front has taken off; $ct=L$, and so

$$t=\frac{L}{c}. \tag{5}$$

During this time, the front of the stick will have traveled a distance of

$$vt=\frac{vL}{c}. \tag{6}$$

It is clear from this perspective that overextension occurs for all velocities—it will always take some time for the information to reach the back of the stick, and when that information arrives, it needs to really double back on itself because by that time the whole stick has a high velocity and must Lorentz contract; yet it is already longer than L! In this case the coefficient of overextension is

$$\eta_{\bar{E}}=\frac{1}{L}\left(\frac{vL}{c}+L-\frac{L}{\gamma}\right)=\frac{v}{c}-\sqrt{1-\frac{v^2}{c^2}}+1=\frac{\eta_E}{\gamma}. \tag{7}$$

Equation (7) is plotted in Fig. 4.

Suppose again that a stick flies by at high velocity, but this time instead of grabbing the back of the stick you hold out your hand and stop the front with your palm [see Fig. 2(c)]. The information travels through the stick at the speed of light to inform the back of the stick that the front has stopped. This takes a time given by $ct+vt=L/\gamma$, or

$$t=\frac{L}{(c+v)\gamma}, \tag{8}$$

during which the back of the stick travels a distance

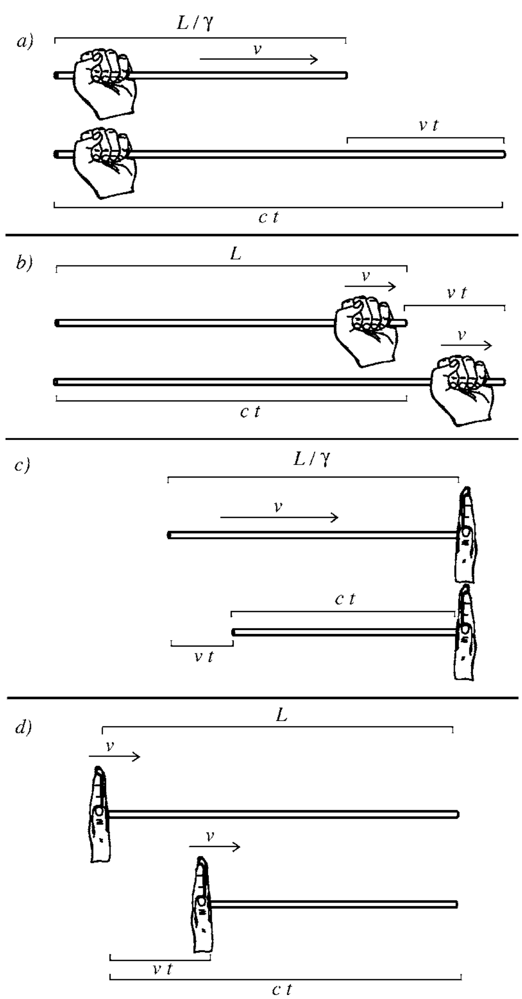

Fig. 2. The four situations. (a) The stick is initially flying past with velocity v and the hand grabs the back of the stick, stopping it immediately. (b) The stick is initially at rest and the hand accelerates the front end of it. (c) The stick is initially in motion and the hand stops it head on. (d) The stick is initially at rest and the hand strikes the back end of it.

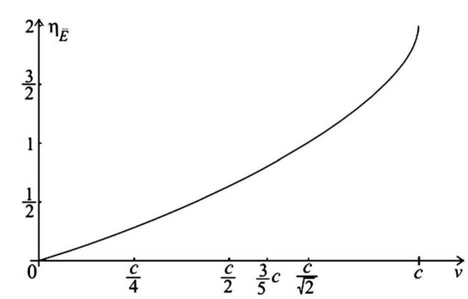

Fig. 4. Plot of the overextension coefficient in Eq. (7). $\eta_{\bar{E}}$ reaches an upper limit of 2 as v approaches c, which corresponds to a maximum length of twice the proper length. $\eta_{\bar{E}}=1$ when $v=c/\sqrt{2}$, which corresponds to a maximum length of $(1+1/\sqrt{2})$ times the proper length.

Fig. 5. Plot of the overcompression coefficient in Eq. (11). η_C reaches an upper limit of 1 as v approaches c, which corresponds to a minimum length of 0. $\eta_C=1/2$ when $v=(3/5)c$, which corresponds to a minimum length of half the proper length.

$$vt = \frac{vL}{(c+v)\gamma}. \tag{9}$$

This time the stick is now even shorter than L/γ when it must eventually extend to L—it is overcompressed. It is clear from this perspective that overcompression occurs for all velocities.

We define the coefficient of overcompression to be proportional to the length it will eventually assume minus the minimum length,

$$\eta_C \equiv \frac{1}{L}\left[\frac{L}{\gamma_f} - \left(\frac{L}{\gamma_0} - vt\right)\right]. \tag{10}$$

In this case,

$$\eta_C = \frac{1}{L}(L - ct) = 1 - \sqrt{1 - \frac{v^2}{c^2}}\frac{c}{c+v} = \frac{\eta_E}{D}. \tag{11}$$

Equation (11) is plotted in Fig. 5.

Finally, instead of pulling the front end to accelerate the stick, we push on the back end [see Fig. 2(d)]. The time it takes for the information to reach the front of the stick is given by $ct=L$, or

$$t = \frac{L}{c}. \tag{12}$$

In this time the back of the stick travels a distance

$$vt = \frac{vL}{c}, \tag{13}$$

and the overcompression coefficient is

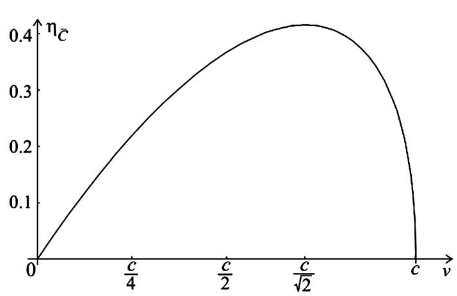

Fig. 6. Plot of the overcompression coefficient in Eq. (14). $\eta_{\bar{C}}$ reaches a maximum of $\sqrt{2}-1$ when $v=c/\sqrt{2}$, which corresponds to a minimum length of $(1-1/\sqrt{2})$ times the proper length.

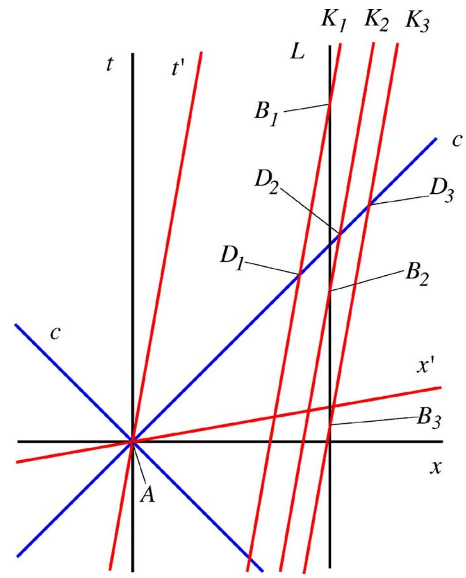

Fig. 7. (Color online) Lock and key paradox: Initial reference frames. The x,t coordinates are the lock and the x',t' coordinates are the key; $x=0$ is the left end of the lock (a and b in Fig. 1); $x=L$ is the position of the button (l in Fig. 1); and $x'=0$ (the t' axis) is the back of the key. The lines K_1, K_2, and K_3 represent the front end of the key for three different lengths. A is the point in space-time when the back of the key hits the prongs of the lock. The speed of light has been set to 1 for clarity and two lines represent the light cone for the point A—the signal carrying the news that the rear end of the key has hit the lock. D_1, D_2, and D_3 (not related to the relativistic Doppler shift factor D) represent the points in space-time when the information reaches the tips of the keys, K_1, K_2, and K_3 respectively; B_1, B_2, and B_3 represent the points in space-time when the front ends of the keys would reach the button if the key kept on going at its initial speed.

$$\eta_{\bar{C}} = \frac{1}{L}\left(\frac{L}{\gamma} - \left(L - \frac{vL}{c}\right)\right) = \sqrt{1 - \frac{v^2}{c^2}} + \frac{v}{c} - 1 = \frac{\eta_C}{\gamma}. \tag{14}$$

Equation (14) is plotted in Fig. 6.

III. RESOLVING THE LOCK AND KEY PARADOX

It remains to be shown that overextension resolves the lock and key paradox; that is, if the velocity is sufficient for the key to hit the button in the key's frame, then overextension will make it hit the button in the lock's frame. One way to show this is with a Minkowski diagram. Figure 7 shows the initial frame of the key passing by the rest frame of the lock. If this paradox could be resolved by issues concerning simultaneity alone, like the barn and ladder paradox, the story would end with Fig. 7.

Three possible lengths are included in Fig. 7 as representative of three length regimes. K_1 receives the information D_1 before it reaches the button B_1—therefore, in a diagram that includes the collision (Fig. 8) K_1 will never reach the button. K_2 reaches the button B_2 before the information gets to the tip D_2. K_3 reaches the button B_3 before the back end even hits the prongs of the lock (in the key's frame).

Now, let's introduce the collision into the diagram. For simplicity, we assume that the lock is so heavy that it doesn't budge when the lock and key collide. This simplification weakens the overextension effect, so if we can still manage to resolve the paradox with this restriction, then we can resolve the paradox without it.

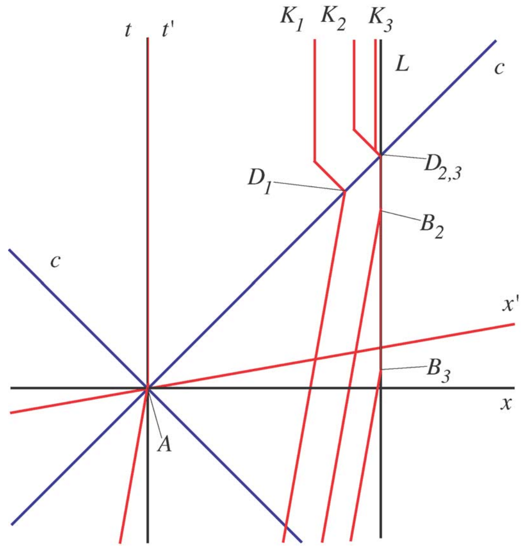

Fig. 8. (Color online) Effect of stopping the key. The collision is included. At all Ds, the Ks immediately assume the speed of light and contract until they reach their proper length. The t' axis becomes the t axis as soon as A occurs.

Figure 8 represents the collision in the Minkowski diagram. K_1 never reaches the button, but both K_2 and K_3 do, even though if they were at rest they would be too short. In Fig. 8, D_1 represents the point when the sound reaches K_1, and $D_{2,3}$ represents the point when the sound reaches K_2 and K_3, which are meanwhile waiting idly by the button that they have pressed. In the initial paradox, when the information travel limits are not considered, only K_3 would have pressed the button. The tacit assumption was that information is communicated instantly. Yet both K_2 and K_3 press the button when the signal travel time is taken into account. Evidently the overextension effects are stronger than the Lorentz contraction effects. Overextension resolves the lock and key paradox.

Finally, let's demonstrate algebraically that overextension resolves the paradox. The smallest possible key length K that still reaches the button by Lorentz contraction, according to the key, is L/γ (where L is the depth of the lock). The smallest K that reaches the button by overextension, according to the lock, is found by

$$\frac{K}{\gamma} + \frac{vK}{\gamma(c-v)} = L, \tag{15}$$

which simplifies to

$$K = L\gamma\left(1 - \frac{v}{c}\right), \tag{16}$$

which can be rewritten as

$$K = \frac{1}{1 + \frac{v}{c}} \frac{L}{\gamma} \leq \frac{L}{\gamma}. \tag{17}$$

The minimum length of the key required to reach the button due to overextension is less than the minimum length required to reach the button due to Lorentz contraction. If the velocity is sufficient for the key to hit the button in the key's frame according to Lorentz contraction, then overextension will make it hit the button in the lock's frame. Overextension resolves the lock and key paradox.

As a final note, overextension has resolved the Lorentz contraction paradox, but perhaps it has created its own, even more devious paradox. That is: for some velocity the key would overextend enough to reach the button according to the lock; but according to the key, the key would not overextend enough to reach the button, or vice versa. That is, the situation can be examined from either the lock's frame [which we have done and refers to the situation in Fig. 2(a)] or the key's frame [which refers to the situation in Fig. 2(b)]. Because the two frames rely on different equations, perhaps they have different minimum values of K. This is not the case. The smallest K that reaches the button by overextension according to the lock [by η_E, Eq. (4), shown explicitly in Eq. (15)] is the same as the smallest K that reaches the button according to the key [by $\eta_{\bar{E}}$, Eq. (7)]. That is,

$$L\gamma\left(1 - \frac{v}{c}\right) = \frac{L}{\gamma}\frac{1}{1 + \frac{v}{c}}, \tag{18}$$

where the right-hand side is the smallest K that reaches the button by overextension according to the key.

Furthermore, suppose we make the key unbudgingly heavy and allow the lock to overcompress to hit the button. The minimum length of the lock in this case is found to be equal to the minimum length of the key found in Eq. (18). Overextension resolves the lock and key paradox and does so without leaving new holes.

IV. DISCUSSION

What does all this say about the limits of rigidity in special relativity? There is no such thing as a rigid object in special relativity if rigidity is assumed to require a constant distance between any two points on a body. The material we considered here is as rigid as is allowable by relativity. It is unbreakable and sound travels at the speed of light. It is the theoretical limit of rigidity; the effects we have studied have nothing to do with the elasticity of the material. These are effects that are required by the laws of special relativity alone. Overextension and overcompression reveal something not about the material, but about special relativity and spacetime.

The "elasticity" of otherwise rigid bodies in a relativistic context has been examined before,[2–4] although the literature is incomplete. My hope is that this paradox can raise new interest in the conceptually rich, but often neglected, nature of rigidity and elasticity in special relativity.

I cannot shed any more light as to why the relativistic doppler shift factor D fits so elegantly into the equations, and whether or not it hints at deeper connections waiting to be made. This paper has only scratched the surface of this relativistic elasticity, working only in one dimension.

A simple variant on the calculations here would be to consider a slower speed of the propagation of stress along the material. Doing so would incorporate special relativistic velocity addition into the solution and would magnify the overextension/overcompression effects found by the calculations used in this paper. It also focuses more on the material in question, and in so doing draws away from the elasticity due to the limits of information travel time required by special relativity.

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