Projectiles, pendula, and special relativity

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The kind of flat-earth gravity used in introductory physics appears in an accelerated reference system in special relativity. From this viewpoint, we work out the special relativistic description of a ballistic projectile and a simple pendulum, two examples of simple motion driven by earth-surface gravity. The analysis uses only the basic mathematical tools of special relativity typical of a first-year university course. © 2005 American Association of Physics Teachers. [DOI: 10.1119/1.1862632]

I. INTRODUCTION

Students often see special relativity as having an inherent elegance, but see no overlap with the problems they study in an introductory physics course. To tie the two student experiences closer together, this paper presents a relativistic treatment of two of the simpler problems studied in elementary mechanics courses: ballistic motion and the simple pendulum.

These familiar examples share an important feature: in both problems, the driving force is the weight of gravity. This feature opens the possibility of treating these problems relativistically by elementary means. This possibility arises due to the equivalence principle. All objects have the same acceleration in a gravitational field; a rock and a feather fall at the same rate in a gravitational field. Consequently, objects in a freely falling frame (like a falling elevator) have no apparent weight.

In relativistic gravitation theories, like Einstein's general relativity, the interpretation of the equivalence principle goes in the opposite direction. In these theories it is the freely falling frame that is the inertial frame. In a frame that is not freely falling, an apparent weight force arises. But it is a pseudoforce, a fictitious force like the centrifugal force, and is an artifact of the noninertial reference frame.^{1,2} Thus a downward apparent weight force is perceived on the surface of the earth because the surface of the earth is accelerating upward at 9.81 m/s² with respect to a freely falling frame.

Here we exploit this point of view. To study ballistic motion and pendula, we do not need a complete relativistic theory of gravity; we need only to account for the weight force (or pseudoforce). We work completely in gravity-free special relativity, and we introduce weight by working in an upward accelerating special relativistic reference frame.

This article is intended for students who have had only the beginnings of relativity. All that is required is the simple (one-dimensional) Lorentz transform, the concept of proper time along a worldline, the acceleration four-vector, and the fact that its components in different reference frames are related by the Lorentz transformation. For more advanced students, the results may still be of interest, and are more easily derived with techniques like covariant differentiation. For such students, a more compact presentation, using more advanced techniques, is given in Appendix A. In order to give a more accessible description, some details are relegated to the endnotes.

II. EARTHLIKE FRAME

We may take the relativistic point of view in introductory Newtonian physics by invoking a freely falling t, x, y, z reference frame near the surface of a flat earth, and by considering this reference frame to be an inertial frame. In this inertial frame, the x, y, z spatial coordinates constitute a Cartesian spatial grid, and t is the universal time of Newtonian physics.

We then imagine a swarm of tiny rocket ships accelerating upward with respect to the inertial frame. (See Fig. 1). At t = 0 each of the rockets is (momentarily) at rest. We assign labels $\tilde{x}, \tilde{y}, \tilde{z}$ to the rockets, such that at t=0 we have $\tilde{x}=x$, $\tilde{y}=y$, and $\tilde{z}=z$. Because the rockets move in the *x* direction, they maintain $\tilde{y}=y$ and $\tilde{z}=z$, but *x* for each rocket is a function of time and it is this time dependence that makes our rocket-borne reference frame an accelerated, and therefore noninertial, frame with a weight force like that of the earth's surface.

We clearly want to have our rocket-borne reference frame to be accelerating upward. In Newtonian mechanics this acceleration would be done by choosing

 $x = \tilde{x} + \frac{1}{2}gt^2. \tag{1}$

With this choice each rocket moves in the same way and the spatial $\tilde{x}, \tilde{y}, \tilde{z}$ grid is always a Cartesian system for measuring distances. Newton's second law can be used in this noninertial system if every mass element *m* is taken to have a gravitational (pseudo)force *mg* acting on it in the negative \tilde{x} direction. We can do mechanics either in the freely falling inertial frame with no gravity or in the noninertial frame with gravity.

In special relativity the equivalent construction has some new subtleties. Our primary reference frame t,x,y,z is now a Minkowski coordinate system, with no gravity. As in the Newtonian case, we again invoke the swarm of rockets that are momentarily at rest in the t,x,y,z frame at t=0. Again we assign labels to the rockets such that $\tilde{y}=y$ and $\tilde{z}=z$ for all t. We must now choose how the rockets are to accelerate, that is, we must specify x(t) for each rocket.

It turns out that the x(t) in Eq. (1) is not ideal in special relativity. The reference frame created by that x(t) would have undesirable features. For example, a rocketeer might want to measure the distance from her rocket to another nearby rocket. This measurement would be done in her momentarily comoving frame. The distance measured in this



Fig. 1. The earthlike $\tilde{x}, \tilde{y}, \tilde{z}$ spatial coordinate system is kept stationary with respect to the earth by the thrust provided by the rocket engines. The inertial x, y, z coordinate system is freely falling.

way would change in time. The rocket frame, then, would not be an unchanging frame like the reference frame used in introductory Newtonian mechanics.

The choice of x(t) that is close to ideal turns out to be³

$$x^2 - c^2 t^2 = \kappa^2,$$
 (2)

where κ is a constant. The reason for favoring this choice is not obvious, but at least one of its features is comforting. For $ct \ll \kappa$, Eq. (2) becomes

$$x \approx \kappa + \frac{1}{2} \frac{c^2}{\kappa} t^2.$$
(3)

Thus, when a rocket is moving at nonrelativistic velocity $(dx/dt \approx c^2 t/\kappa \ll c)$, this relativistic choice of x(t) takes the Newtonian form in Eq. (1) if we take the acceleration to be c^2/κ .

The constant κ in Eq. (2) can be different for each rocket, as in Eq. (1), so that this constant can be used to assign an \tilde{x} coordinate to each rocket. As we shall demonstrate, it is best to do this by choosing⁴

$$x^2 - c^2 t^2 = (\tilde{x} + c^2/g)^2.$$
(4)

The meaning of \tilde{x} is potentially confusing. It is a constant along the world line of any particular rocket. But we will also use it as a spatial label in the \tilde{x} , \tilde{y} , \tilde{z} system. A particle—like a ballistic projectile or a pendulum bob moving from one rocket location to another would have a time varying value of \tilde{x} , and it is meaningful to consider $\tilde{x}(t)$ for such a particle.

The $\tilde{x}, \tilde{y}, \tilde{z}$ system will be our earthlike system. It is to be considered a spatial reference frame only and is not part of a Minkowski system. To help avoid confusion we will not (except in Appendix A) endow this reference frame with an associated time coordinate. Rather, we will discuss the dynamics of particles (ballistic projectiles and pendulum bobs) with the proper time τ for those particles, the time measured by clocks carried on the particles.

The real justification for Eq. (4) is that with this choice, the rocket-borne $\tilde{x}, \tilde{y}, \tilde{z}$ reference frame has three important properties that qualify it as an earthlike spatial reference frame. These properties are best understood with a spacetime diagram like that in Fig. 2. The diagram shows worldlines for two arbitrary rockets, labeled 1 and 2. According to Eq. (4) these worldlines are hyperbolae asymptotic to x = ct. In this diagram P_1 (coordinates t_1, x_1) is an event on the worldline of rocket 1, and t', x', y', z' is a Minkowski coordinate system instantaneously comoving with rocket 1 at event P_1 . Event P_2 (coordinates t_2, x_2) is the event on the worldline of rocket 2 that is simultaneous, in the instantaneously comov-



Fig. 2. In a t,x spacetime diagram, the worldlines of rockets 1 and 2 are shown, along with the coordinate axes of the t',x' Minkowski reference frame, the frame comoving with rocket 1 at event P_1 .

ing frame, with P_1 . That is, events P_1 and P_2 have the same value of t', and are simultaneous as seen in the reference frame of P_1 .

The three special properties of the earthlike system include the following. (i) In a frame that is instantaneously comoving with one rocket, all rockets are instantaneously at rest. (In other words, the velocity dx/dt of rocket 2 at P_2 is the same as that of rocket 1 at P_1 .) (ii) The acceleration of each rocket is a constant in time. (iii) The distance between any two rockets, as measured in an instantaneously comoving frame, is time independent. These properties, proved in Appendix B, establish that the rocket-borne $\tilde{x}, \tilde{y}, \tilde{z}$ system is a "rigid" framework for spatial measurements. Although it is not part of an inertial reference frame, it is, in a sense, the same at all times.

III. BALLISTIC TRAJECTORIES

A ballistic projectile has only the weight force acting on it. This means that its worldline will be straight in a Minkowski coordinate system.⁵ With little loss in generality we choose the projectile to be moving in the *x*, *y* plane, with *x* having the fixed value $x = c^2/g$, so that the projectile starts off with $\tilde{x}=0$ at t=0. In the freely falling t, x, y, z frame, the projectile is moving only in the *y* direction, and we specify its motion by $y = vt \equiv c\beta t$. A moment of proper time $d\tau$ and of coordinate time dt are related, as usual, by $d\tau = \sqrt{(1-\beta^2)}dt \equiv dt/\gamma$.

We choose τ to be zero when *t* is zero, and the complete description of the projectile motion in the freely falling frame becomes

$$t = \gamma \tau, \quad x = c^2/g, \quad y = c \beta \gamma \tau.$$
 (5)

With Eq. (4), the description in the earth-based frame immediately follows:

$$\widetilde{x} = -c^2/g + \sqrt{(c^2/g)^2 - c^2\gamma^2\tau^2}, \quad \widetilde{y} = c\beta\gamma\tau, \tag{6}$$

and we see that the trajectory has the shape of an ellipse:

$$(\tilde{x} + c^2/g)^2 + \tilde{y}^2/\beta^2 = (c^2/g)^2.$$
(7)

If $v \ll c$ and $\tau \ll c/g \approx 1$ year, then Eq. (6) reduces to

$$\tilde{x} \approx -\frac{1}{2}g\,\tau^2 \approx -\frac{g}{2v^2}\tilde{y}^2, \quad y \approx v\,\tau. \tag{8}$$

It is reassuring that Eq. (6) has the familiar nonrelativistic limit, but it is more interesting in its fully relativistic form.



Fig. 3. Ballistic trajectory for v = 4c/5, starting at $\tau = 0$ and ending at $\tau = (\frac{3}{5})(c/g)$.

Figure 3 shows the elliptical trajectory for the case v = 4c/5. This ellipse differs noticeably from a parabola. It is of interest that the trajectory ends at the point $\tilde{x} = -c^2/g$, $\tilde{y} = (4/5)c^2/g$, corresponding to the proper time $\tau = (3/5)c/g$. This sudden end is not an indication of a dramatic physical event. Rather, it signals the limit of the ability of the $\tilde{x}, \tilde{y}, \tilde{z}$ coordinates to cover the spacetime.⁶

IV. SIMPLE PENDULUM

The key idea in understanding the pendulum is the requirement that the motion is determined by constraining forces in the $\tilde{x}, \tilde{y}, \tilde{z}$ frame. In the specific case of pendulum motion, the constraint is that the pendulum bob move in a circular arc. We will, however, not immediately confine ourselves to the case of a circular-motion pendulum, but will keep the description as general as possible for as long as possible. Initially we will suppose only that the motion can be described by two functions $\tilde{x}(\tau), \tilde{y}(\tau)$.⁷ The constraint on the motion (for example, that the particle move in an arc of radius *L*) can be thought of as a curve in the $\tilde{x}-\tilde{y}$ plane.

We are faced now with the task of combining a description of a constrained path in the earthlike $\tilde{x}, \tilde{y}, \tilde{z}$ frame, with an understanding of gravity (there is no gravity) in an inertial frame. To do this, we consider a single moment for the pendulum bob, and invoke the t', x', y', z' Minkowski system that is instantaneously comoving with the earthlike frame.⁸ Because the earthlike frame and the instantaneously comoving frame are momentarily at rest with respect to each other, the spatial directions at that moment are the same in the two frames. Then in the instantaneously comoving frame, we can describe both the constraining path and the (nonexistent) nature of gravity.

We now let

$$U^{t'} \equiv \frac{dt'}{d\tau}, \quad U^{x'} \equiv \frac{dx'}{d\tau}, \quad U^{y'} \equiv \frac{dy'}{d\tau}$$
(9)

be the primed-frame components of the four-velocity of the particle, so that the components of the acceleration fourvector are

$$a^{t'} \equiv \frac{dU^{t'}}{d\tau} = \frac{d^2t'}{d\tau^2}, \quad a^{x'} \equiv \frac{dU^{x'}}{d\tau} = \frac{d^2x'}{d\tau^2},$$
$$a^{y'} \equiv \frac{dU^{y'}}{d\tau} = \frac{d^2y'}{d\tau^2}.$$
(10)

The requirement that there is no acceleration in the direction of motion means that

$$a^{x'}U^{x'} + a^{y'}U^{y'} = 0. (11)$$

For any motion,⁹ the four-acceleration and the four-velocity satisfy

$$c^{2}a^{t'}U^{t'} - a^{x'}U^{x'} - a^{y'}U^{y'} = 0.$$
 (12)

Equations (11) and (12) tell us that $a^{t'}U^{t'}$ must be zero, and hence that $a^{t'}$ must be zero. (The $U^{t'}$ component cannot be zero.)

We take this result as the key to the dynamics: $a^{t'}=0$ in the inertial frame that is momentarily comoving with the earth frame. If the momentary x velocity of the instantaneously comoving (t', x', y', z') frame¹⁰ is $c\beta$ with respect to the x,y,z,t frame, then the Lorentz transformation tells us that

$$a^{t'} = \frac{1}{\sqrt{1 - \beta^2}} (a^t - \beta a^{x/c}).$$
(13)

It is straightforward to show (and is explicitly shown in Appendix B) that the Minkowski frame comoving with the rocket at event t,x has $\beta = ct/x$. The condition in Eq. (13) for no acceleration along the motion becomes

$$0 = a^t - \beta a^x / c = \frac{d^2 t}{d\tau^2} - \frac{t}{x} \frac{d^2 x}{d\tau^2} = \frac{1}{x} \frac{d}{d\tau} \left[x^2 \frac{d}{d\tau} \left(\frac{t}{x} \right) \right].$$
(14)

From Eq. (14) we infer

$$x^{2}\frac{d}{d\tau}\left(\frac{t}{x}\right) = x\frac{dt}{d\tau} - t\frac{dx}{d\tau} = \text{const} \equiv K,$$
(15)

and from Eq. (4) we have

$$x\frac{dx}{d\tau} - c^2 t\frac{dt}{d\tau} = (\tilde{x} + c^2/g)\frac{d\tilde{x}}{d\tau}.$$
(16)

From Eqs. (15) and (16) and from Eq. (4), we can solve for $dx/d\tau$ and $dt/d\tau$ in terms of $d\tilde{x}/d\tau$:

$$\frac{dx}{d\tau} = \frac{c^2 t K}{\left(\tilde{x} + c^2/g\right)^2} + \frac{x}{\left(\tilde{x} + c^2/g\right)} \frac{d\tilde{x}}{d\tau},\tag{17}$$

$$\frac{dt}{d\tau} = \frac{xK}{(\tilde{x} + c^2/g)^2} + \frac{t}{(\tilde{x} + c^2/g)} \frac{d\tilde{x}}{d\tau}.$$
(18)

The differential of proper time $d\tau$ along the worldline of the pendulum bob, in terms of differentials of the inertial coordinates, is

$$(d\tau)^2 = (dt)^2 - c^{-2}(dx)^2 - c^{-2}(dy)^2.$$
⁽¹⁹⁾

We now substitute $dy = d\tilde{y}$ and the results in Eqs. (17) and (18) into Eq. (19) to arrive at an expression for the motion entirely in terms of $\tilde{x}(\tau)$ and $\tilde{y}(\tau)$:

$$\left(\tilde{x} + \frac{c^2}{g}\right)\sqrt{c^2 + \left(\frac{d\tilde{x}}{d\tau}\right)^2 + \left(\frac{d\tilde{y}}{d\tau}\right)^2} = \text{const.}$$
(20)

Equation (20) is a single equation for the two functions $\tilde{x}(\tau), \tilde{y}(\tau)$. This equation gives us the basis for describing whatever constrained motion we wish, once we have chosen the path—the relationship between \tilde{x} and \tilde{y} imposed by the constraint. There can certainly be different opinions about



Fig. 4. A pendulum making an arc of a circle in the \tilde{x}, \tilde{y} plane.

what constitutes the path of a pendulum bob in relativity, but the choice made here is the most obvious. As pictured in Fig. 4, the pendulum bob maintains a distance *L* from the pivot, as measured in the $\tilde{x}, \tilde{y}, \tilde{z}$ frame. With the pivot at the \tilde{x}, \tilde{y} origin, the constraint is that $\tilde{x}^2 + \tilde{y}^2 = L^2$. This constraint can be written as

$$\tilde{x} = -L\cos\theta(\tau), \quad \tilde{y} = L\sin\theta(\tau),$$
(21)

where θ is the angle shown in Fig. 4.¹¹ In terms of $\theta(\tau)$, Eq. (20) takes the form

$$\left(-L\cos\theta + \frac{c^2}{g}\right)\sqrt{c^2 + L^2 \left(\frac{d\theta}{d\tau}\right)^2} = \text{const} = c \left(-L\cos\theta_{\text{max}} + \frac{c^2}{g}\right), \qquad (22)$$

where we have defined θ_{max} as the maximum angular excursion of the pendulum, the angle at which $d\theta/d\tau=0$.

We can now solve for $d\tau/d\theta$ and integrate to find the length of proper time for a quarter of a period *P*,

$$\frac{P}{4} = \int_{0}^{\theta_{\text{max}}} \left(\frac{d\tau}{d\theta}\right) d\theta$$
$$= \frac{L}{c} \int_{0}^{\theta_{\text{max}}} \left[\left(\frac{-L\cos\theta_{\text{max}} + c^{2}/g}{-L\cos\theta + c^{2}/g}\right)^{2} - 1 \right]^{-1/2} d\theta. \quad (23)$$

The value of P given by Eq. (23) is smaller than the standard small angle period

$$P_0 = 2\pi \sqrt{\frac{L}{g}}.$$
(24)

In Fig. 5, the ratio P/P_0 is plotted as a function of Lg/c^2 for the case $\theta_{\text{max}} = 5^\circ$. For extremely long *L*, comparable to $c^2/g \approx 10^{16}$ m, the reduction is very significant. And this reduction cannot be ascribed simply to the slowing of proper time for a rapidly moving object. For example, the maximum value of $v \equiv Ld\theta/d\tau$ is $\approx 0.175 c$ for $\theta_{\text{max}} = 5^\circ$ and Lg/c^2



Fig. 5. The reduction of the period as a function of Lg/c^2 for $\theta_{\text{max}}=5^\circ$.

=0.8, corresponding to a time dilation factor of $\sqrt{1-v^2/c^2} \approx 0.985$. The reduction shown in Fig. 5 is much greater than this.

V. CONCLUSIONS

"Special relativistic gravity" is a fictitious force arising in a noninertial earthlike reference frame. We have shown that ballistic and pendulum motions can be analyzed in this frame by using the principle that there is no gravity in a freely falling reference frame. This analysis makes good pedagogical exercises, though of considerably different difficulty. The study of ballistic motion is simple, while that of pendulum motion brings in more physical ideas and somewhat trickier mathematics.

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APPENDIX A: CALCULATIONS WITH GEODESICS AND COVARIANT DIFFERENTIATION

Here we exploit more advanced mathematical techniques to simplify the calculations we discussed in the main text. For simplicity we use units in which c=1. We can describe the relationship of the inertial x,y,z system to the earthbased $\tilde{x}, \tilde{y}, \tilde{z}$ system with the transformation

$$t = (\tilde{x} + 1/g)\sinh g\,\tilde{t},\tag{A1a}$$

$$x = (\tilde{x} + 1/g)\cosh g \tilde{t}, \tag{A1b}$$

$$y = \tilde{y},$$
 (A1c)

$$z = \tilde{z}$$
. (A1d)

In the noninertial $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}$ coordinates, the metric takes the form known as the Rindler geometry:^{12,13}

$$ds^{2} = -(1+g\tilde{x})^{2}dt^{2} + d\tilde{x}^{2} + d\tilde{y}^{2} + d\tilde{z}^{2}.$$
 (A2)

A ballistic trajectory is a geodesic worldline, so fourvelocity components must satisfy the geodesic equation $DU^{\alpha}/d\tau=0$. From this it is easy to show that $U^{\tilde{y}}$ and $U_{\tilde{t}}$ are constant, and we choose the constants to be denoted by u and E, respectively. The fact that $\vec{U} \cdot \vec{U} = -1$ implies that

$$1 = \frac{(U_0)^2}{(1+g\tilde{x})^2} - (U^{\tilde{y}})^2 - (U^{\tilde{x}})^2$$
$$= \frac{E^2}{(1+g\tilde{x})^2} - u^2 - \left(\frac{d\tilde{x}}{d\tau}\right)^2.$$
(A3)

If we choose \tilde{x} to be zero when $d\tilde{x}/d\tau$ is zero, then $E^2 = 1 + u^2$. The resulting differential equation is

$$\left(\frac{d\tilde{x}}{d\tau}\right)^2 = (1+u^2) \left[\frac{1}{(1+g\tilde{x})^2} - 1\right].$$
 (A4)

It is easy to check that $\tilde{x}(\tau)$ given by Eq. (6) is the solution to this differential equation for $\tilde{x}=0$ when $\tau=0$.

For the motion of the pendulum bob, we use the fact that in the "stationary" $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}$, coordinates, the acceleration of the bob must have no component along the motion, or

$$a^{\tilde{x}}U^{\tilde{x}} + a^{\tilde{y}}U^{\tilde{y}} = 0.$$
(A5)

Because $\vec{a} \cdot \vec{U} = 0$, it follows that $a^t = 0$, or, equivalently, $a_{\tilde{t}} = 0$. By using covariant differentiation in the $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}$ coordinates, we have that

$$a_{\tilde{t}} = \frac{dU_{\tilde{t}}}{d\tau} - U_{\tilde{t}}U^{\tilde{x}}\Gamma_{\tilde{t}\tilde{x}} - U_{\tilde{x}}U^{\tilde{t}}\Gamma_{\tilde{t}\tilde{t}} = 0.$$
(A6)

It is straightforward to check that the Christoffel terms cancel each other, so we can conclude that $U_{\tilde{t}}$ is a constant.¹⁴ From $\vec{U} \cdot \vec{U} = -1$, we have then that

$$1 = \frac{(U_{\tilde{t}})^2}{(1+g\tilde{x})^2} - (U^{\tilde{y}})^2 (U^{\tilde{x}})^2,$$
(A7)

or

$$(1+g\tilde{x})^2 \left[1+\left(\frac{d\tilde{x}}{d\tau}\right)^2+\left(\frac{d\tilde{y}}{d\tau}\right)^2\right]=\text{const},$$
 (A8)

which is identical to Eq. (20).

APPENDIX B: THE THREE SPECIAL PROPERTIES OF THE EARTHLIKE SYSTEM

Brief derivations are given here of the three properties stated in Sec. II. The derivations will make use of the Minkowski coordinate system t', x', y', z' in the reference frame that is comoving with rocket 1 at the point P_1 (coordinates t_1, x_1) on the rocket's worldline, as shown in Fig. 2. The idea is that $\tilde{x}, \tilde{y}, \tilde{z}$ are not part of a Minkowski coordinate system, so we cannot directly apply to it simple Lorentz transformations. But we *can* apply the mathematics of Minkowski systems to t', x', y', z'.

For convenience we will hide some bothersome factors of c by introducing the common notation of a 0 coordinate $x^0 \equiv ct$, and a 0 component of four-vectors, such as $U^0 \equiv cU^t$ for the time component of the four-velocity.

1. Simultaneity of rocket speed

We first show that any rocket "sees" all other rockets to be at rest with respect to itself. More specifically, we will show that at a given moment of t', all rockets have the same speed with respect to the t, x, y, z Minkowski reference frame (and hence with respect to any Minkowski reference frame). We start by noticing that in Fig. 2 both P_1 and P_2 lie on the x' axis, the set of events with the same value of t'. The equation for that axis is dt' = 0. From the Lorentz transformation between the primed and unprimed system this condition gives us

$$c\,dt - \beta dx = 0,\tag{B1}$$

as the equation for the axis, where $c\beta$ is the speed of the primed frame with respect to the unprimed frame. The value of $c\beta$ is simply dx/dt for worldline 1 at point P_1 . From Eq. (4), $c\beta = dx/dt = c^2t_1/x_1$, where t_1, x_1 are the coordinates of P_1 . We can now combine this result with Eq. (B1) to find that the slope of the x' axis is

$$\frac{dt}{dx} = \frac{t_1}{x_1}.$$
(B2)

Because the axis must go through the point t_1, x_1 , it follows that the equation of the axis is

$$\frac{t}{x} = \frac{t_1}{x_1}.$$
(B3)

The x' axis is then simply the line going through the t,x origin and through P_1 .

Clearly we would have obtained precisely the same line if we had started with point P_2 . Thus the x' axis would be the same for the Minkowski coordinate system comoving with the rocket at point P_2 . But if the x' axis is the same, then the reference frame comoving with rocket 1 at P_1 is the same as the reference frame comoving with rocket 2 at P_2 . In other words, the speed of rocket 1 at P_1 is the same as that of rocket 2 at the same moment of comoving time.

2. Rocket acceleration

From Eq. (4) we have that along a rocket worldline $dx/dt = c^2 t/x$, and hence the Lorentz factor is

$$\gamma = 1/\sqrt{1 - (dx/cdt)^2} = \frac{x}{\tilde{x} + c^2/g}.$$
 (B4)

The components of the four-velocity and four-acceleration are

$$U^{0} = c \frac{dt}{d\tau} = c \gamma = \frac{cx}{\tilde{x} + c^{2}/g}, \quad U^{x} = \frac{dt}{d\tau} \frac{dx}{dt} = \frac{c^{2}t}{\tilde{x} + c^{2}/g},$$
(B5)

and

$$a^{0} = \gamma \frac{c dx/dt}{\tilde{x} + c^{2}/g} = \frac{c^{3}t}{(\tilde{x} + c^{2}/g)^{2}},$$

$$a^{x} = \gamma \frac{c^{2}}{\tilde{x} + c^{2}/g} = \frac{c^{2}x}{(\tilde{x} + c^{2}/g)^{2}}.$$
 (B6)

The quantity $a \equiv \sqrt{\vec{a} \cdot \vec{a}}$ is an invariant that signifies the acceleration "felt" by each rocket. (It is, for example, the component $a^{x'}$ of the acceleration, when evaluated in an instantaneously comoving Minkowski reference frame.) With the above results we can evaluate *a* to be

$$\sqrt{(a^x)^2 - (a^0)^2} = \frac{c^2}{\tilde{x} + c^2/g}.$$
 (B7)

The scalar *a* is then constant along the worldline of each rocket, but varies slightly from rocket to rocket. (For \tilde{x} small compared to 10^{16} m, the variation in *a* is negligible.)

3. Rigidity of the earthlike frame

The third important property of the earthlike reference frame is its spatial rigidity, the time independence of the separation of the rockets. More precisely, this property is that in a reference frame instantaneously comoving with rocket 1, the distance measured to rocket 2 will be the same at all times; it will not depend on our choice of point P_1 on the worldline.

To prove this we start with the distance as measured in the comoving frame at P_1 . This measurement is simply $x'_2 - x'_1$ made at a single moment of t'. It can be written in the form

$$x_2' - x_1' = \sqrt{(x_2 - x_1)^2 - c^2(t_2 - t_1)^2}.$$
 (B8)

Now $ct_1 = \beta x_1$ and $ct_2 = \beta x_2$ where $c\beta$ is the speed with respect to the *t*,*x*,*y*,*z* system of rocket 1 at point P_1 or of rocket 2 at P_2 . (It was shown above that they are the same.) It follows that the distance is

$$\sqrt{(x_2 - x_1)^2 - c^2(t_2 - t_1)^2} = \gamma^{-1}(x_2 - x_1).$$
(B9)

We can next use $x_2 = \gamma(\widetilde{x_2} + c^2/g)$ and $x_1 = \gamma(\widetilde{x_1} + c^2/g)$ to write the result as

distance =
$$\widetilde{x_2} - \widetilde{x_1}$$
. (B10)

This completes the proof that the distances separating rockets are constant in time.

¹Edwin F. Taylor and John A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966), Sec. 2.

²Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Secs. 6.2 and 6.3.

³A detailed discussion of the transformation in Eq. (4) and the relation to an accelerated spatial reference frame can be found in E. A. Desloge and R. J. Philpott, "Uniformly accelerated reference frames in special relativity," Am. J. Phys. **55**, 252–261 (1987), and in E. A. Desloge, "Spatial geometry in a uniformly accelerating reference frame," *ibid.* **57**, 598–602 (1989). ⁴The notation here is slightly different from that used by Desloge.³

The symbol g used there and that used here are related by $g_{\text{desloge}} = c^2/(\bar{x} + c^2/g_{\text{here}})$.

- ⁵There is more than one way of interpreting the gravitational force. I. R. Lapidus has studied ballistic motion very differently, treating gravity as a velocity-dependent four-force, in "Motion of a relativistic particle acted upon a constant force and a uniform gravitational field," Am. J. Phys. **40**, 984–988 (1972).
- ⁶The range of x, t to which Eq. (4) applies is limited to $|x| \ge ct$, but there are no real boundaries to the spacetime at $x = \pm ct$. These are boundaries only of the regions that are covered by \tilde{x} values in Eq. (4).

⁷It is only for convenience that we are assuming that the motion is planar. The analysis to follow is very easily extended to three-dimensional motion by adding a $(dz)^2$ term to Eq. (19).

⁸Note that the comoving frame is not instantaneously comoving with the pendulum bob. Rather, at the moment in question it is comoving with the rocket at the location of the pendulum bob. By the special properties of the worldlines of Eq. (4) it also is instantaneously comoving with all other rockets.

⁹The orthogonality of the four-velocity and four-acceleration also is standard. See, for example, Ref. 2, Eq. (6.2). It follows immediately from differentiating $c^2U'^2 - U^{x'2} - U^{y'2} = c^2$ with respect to τ .

¹⁰The velocity parameter β here has no direct relation to the velocity parameter β used in Sec. III.

- ¹¹The variable θ is illustrated by, but not defined by, Fig. 4. The definition is $\theta \equiv \tan^{-1}(-\tilde{y}/\tilde{x})$.
- ¹²Discussions of this metric can be found in Ref. 2, Sec. 6.2; W. Rindler, "Kruskal space and the uniformly accelerated frame," Am. J. Phys. 34, 1174–1178 (1976); A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton U. P., Princeton, 1975), Prob. 1.17; G. F. R. Ellis and R. M. Williams, *Flat and Curved Spacetime* (Oxford U. P., Oxford, 2000), pp. 169–178.
- ¹³The fact that the spatial part of this metric is independent of time is related to the point made in Sec. II that the spatial distances between rockets, measured in an instantaneously comoving frame, do not change in time.
- ¹⁴It is well known that for a metric with coefficients independent of the time coordinate, the covariant component U_i is constant along a geodesic. See, for example, Ref. 2, Sec. 25.2. This quantity can be considered a mechanical energy (kinetic plus gravitational) that is conserved for the motion. The metric in Eq. (A2) is independent of \tilde{t} , but, due to the constraining forces, the motion of the pendulum bob is not geodesic. The fact that $U_{\tilde{t}}$ is conserved may be interpreted to mean that, as in Newtonian physics, the mechanical energy is conserved because the constraining forces do no work. For a generalization of constraining forces and conserved mechanical energy to any stationary spacetime see R. H. Price, "Normal forces in stationary spacetimes," Gen. Relativ. Gravit. **36**, 2171–2173 (2004).